Efficient Firm Dynamics
in a Frictional Labor Market *

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Abstract

The introduction of large firms into labor search models raises the question how wages are set when average and marginal product differ. We propose an alternative to the existing bargaining models by allowing firms to compete for labor. Fast growing firms do not only post more vacancies, but they also post higher wages to attract more workers. Therefore, they fill each vacancy with higher probability, which is consistent with empirical regularities. Qualitatively the model also captures most other empirical regularities about firm size, job creation, and pay. In contrast to existing bargaining models that always induce inefficiencies on the intensive hiring margin, these factual implications of our model for firm dynamics are indeed socially optimal. Social efficiency obtains on extensive and intensive margins of job creation and job destruction, both with idiosyncratic and with aggregate productivity shocks. Moreover, the planner solution allows for a tractable characterization which is useful for computational applications.

JEL classification: E24; J64; L11
Keywords: Labor market search, firm-size distribution, multi-worker firms, job creation and job destruction

PRELIMINARY VERSION
MARCH 4, 2010

1 Introduction

Search models of the labor market have traditionally treated the production side very simplistically: Each firm wants to hire one worker, which is usually equivalent to having several workers with constant marginal product (see e.g. the surveys by Mortensen and Pissarides (1999) and Rogerson, Shimer, and Wright (2005)). While successful in many dimensions, these models are silent about all aspects that relate to employer size, even though firm size and firm dynamics are important for both wages and employment. Larger firms are on average more productive and they tend to pay more (e.g., Brown and Medoff (1989), Oi and Idson (1999)). In percentage terms, older and larger firms grow less but also exit the market less (Evans (1987a, 1987b)). After controlling for worker characteristics, young and fast–growing firms pay higher wages (Brown and Medoff (2003), Belzil (2000)). And larger firms create and destroy more jobs (Davis, Haltiwanger, and Schuh (1996)), they are more sensitive to the business cycle than smaller firms (Moscarini and Postel-Vinay (2009a)), and therefore contribute to aggregate employment dynamics in very different ways. Abstracting from firm size not only misses important labor market facts that we might like to understand, it also limits the ability of labor market models to account for business cycle behavior.

To capture firm size phenomena, a series of recent work has introduced multi-worker firms with decreasing returns to labor into standard labor search models, considering among others the implications for wages and unemployment (e.g., Bertola and Caballero (1994), Smith (1999), Bertola and Garibaldi (2001), Acemoglu and Hawkins (2006), Cahuc, Marque, and Wasmer (2008), Mortensen (2009)), for labor market regulation (Koeniger and Prat (2007)), for business cycles (Elsby and Michaels (2008), Fujita and Nakajima (2009)) and for trade-based employment effects across countries (e.g., Helpman and Itskhoki (2010), Helpman, Itskhoki and Redding (2010a,b)). A central part of all labor market models concerns the wage formation, and all of these contributions rely on a combination of random search together with extensions of the bargaining framework of Stole and Zwiebel (1996). The latter might be viewed as the analogue of standard one-worker-one-firm bargaining for multi-worker settings. The validity of this wage setting assumption has never been analyzed; nonetheless, it dominates the current developments, possibly

due to a lack of alternatives.

One particular concern about the bargaining model with multi-worker firms is that it introduces inefficiencies by assumption. If the marginal product of labor is decreasing, each individual firm hires too many workers (Stole and Zwiebel (1996), Smith (1999), Cahuc, Marque, and Wasmer (2008)).[1] This inefficiency arises before accounting for general equilibrium effects, since the interaction between a single firm and its workers is not even efficient within the match: the firm employs workers up to a point where their marginal product is lower than the hiring cost, since the reduction in marginal product by an additional worker depresses the wages of all inframarginal workers. Such excess employment within the match arises independently of the bargaining power. Inefficiency is therefore an unavoidable feature of this setup, and beneficial government interventions exist by assumption.

Note that this is very different from the standard one-worker-one-firm (or constant returns) setup, where each match (worker-firm-pair) takes decisions that are efficient for the matched parties; whether this leads in equilibrium to efficient job creation and job destruction on the aggregate level depends on the exact bargaining parameter (Hosios (1990)). Therefore, applied work in this area has focused on the planner's solution (see e.g. Merz (1995), Andolfatto (1996), Shimer (2005b)) or compared it with inefficient solutions (e.g. Hall (2005), Hagedorn and Manovskii (2008)) to see which seems to be closer to reality. The current multi-worker setup does not allow for such a horse race between efficient and inefficient outcomes, partly because we do not have plausible search models that justify an analysis of the planner's problem.

The aim of this paper is to propose an alternative wage formation process based on directed search, explore its positive potential for explaining firm dynamics, and to evaluate its implications for efficiency. The main idea behind directed job search is that firms set wages to attract workers rather than to split surplus ex-post. Therefore, firms that have a strong desire to grow do not just post more vacancies, but

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[1]Pissarides (2000) notes that a decreasing marginal product of labor does not affect the results if the production function has constant returns in multiple inputs, and the other inputs such as capital can be adjusted instantaneously. In this case the marginal return of labor, after accounting for adjustment of the other factors, is in fact constant. Cahuc and Wasmer (2001) show that in this case their is no overhiring because the wages of inframarginal workers are not affected. Therefore, the idea of decreasing returns to labor effectively means decreasing returns to scale for a production function with multiple, freely variable inputs.

they also attract more workers for any given vacancy and hence fill jobs at a higher rate. Recent empirical work supports this view: Davis, Faberman, and Haltiwanger (2009) find that firms that double their employment growth rate do so by posting 20% more vacancies (relative to workforce) but they fill a given vacancy with 50% higher probability.[2] While there are also random search models in which job-filling rates differ between firms,[3] this is absent in most papers with bargained wages but constitutes an integral part of competitive search. In such models, firms compete by publicly posting employment contracts and workers react by applying more for attractive contracts. We model both the extensive and intensive margin by allowing entry of new firms and exit of existing firms (extensive margins) and hiring and firing of workers by existing firm (intensive margins). While each firm can post many vacancies at a time, in general it will not immediately jump to its desired size because monetary recruitment costs and human resource costs in hiring and training of prospective employees make such immediate adjustment prohibitively expensive.

An important question is whether our model delivers on the interesting margins of firm dynamics mentioned above that inspired the use of multi-worker firms in the first place. It does. We show analytically that wages are higher in more productive firms and in firms that grow faster. Conditional on firm age, larger firms pay higher wages. Conditional on size, firms that grow faster post more vacancies and fill each vacancy with higher probability.

Furthermore, these factual implications on firm dynamics are indeed socially optimal: A social planner would choose the same path of job flows for each of the firms. Multi-worker firms create and destroy jobs efficiently both on the *extensive* and *intensive* margin, even though bargaining would never lead to efficiency on the latter. The model retains tractability in steady state and even in the presence of idiosyncratic and aggregate shocks, and we show that efficiency arises regardless.

The notion of competitive search builds on existing work with one-worker-per-firm (e.g., Peters (1991), Moen (1997), Acemoglu and Shimer (1999b), Burdett, Shi, and Wright (2001) Shi (2001), Mortensen and Wright (2002), Shimer (2005a)). While these contributions have long established efficiency on the extensive entry margin,

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[2] The numbers refer to firms that grow at 20% instead of 10%, and rely on Figures 5 and 9 in Davis, Faberman, and Haltiwanger (2009).

[3] On this issue, see our discussion of on-the-job search models in the next section.

efficiency on the intensive margin is not obvious: For example Guerrieri (2008) introduces an intensive margin through moral hazard into one-worker-one-firm models and finds efficiency in steady state but not out of steady-state. We show that efficiency on all margins obtains in and out of steady state.[4] Our analysis suggests a plausible environment where the decentralized market achieves efficiency, giving a justification for the study of the planners' solution and its comparisons with other decentralizations. We discuss further related competitive search models such as Garibaldi and Moen (2010) and Hawkins (2006) in detail below.

Our framework is intended as a possible alternative to the existing bargaining framework. The model has the ability to account for important firm dynamics, and therefore can be used to assess their implications for the levels and changes in aggregate employment. And this is achieved without any systematic imperfections in the market mechanism. While it will be a matter of empirical investigation whether alternative wage setting mechanisms lead to inefficient market outcomes, this work shows that fixed setup costs in combination with decreasing returns at the firm level does not need to induce any inefficiencies. Unless wages by themselves are of crucial importance, this model suggests that many of the allocation processes in the labor market might be analyzed as solutions to a planner's allocation.

We also show that the planner's choice of firm dynamics retains tractability because it can be obtained from the solution of a recursive equation at the level of an individual firm in combination with an optimal firm–creation condition. This feature remains true in the presence of aggregate shocks which is particularly helpful for computational applications. In particular, we show that any planner allocation with positive firm entry can be calculated without knowledge of the distribution of employment across existing firms. In this way, our solution method avoids the need for approximation techniques, such as those of Krusell and Smith (1998), that have been applied in the multi-worker search models of Elsby and Michaels (2008) and Fujita and Nakajima (2009) to analyze aggregate labor market dynamics. Technically, our solution method exhibits the feature of "block recursivity" where each agent's value and policy functions are independent of the firm and worker distribution, as in Menzio and Shi (2008).

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[4]On an intuitive level, the main difference to Guerrieri (2008) is that future unemployment rates do not have unpriced externalities on current productivity.

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After a brief review of further related literature, we first present in Section 3 a simplified setup without aggregate or idiosyncratic shocks, except some initial productivity differences and exogenous firm death. This allows for a teachable representation, it establishes the most important insights for dynamics of employment and pay over the life-time of a firm, and it demonstrates the efficiency of the decentralized allocation clearly. In Section 4, we lay out the notationally more complex analysis that takes account of aggregate and idiosyncratic shocks and we characterize the efficient allocation and the decentralized allocation. Since it is not obvious that efficiency extends beyond the stationary environment, we formally prove the equivalence of both of these allocations. Beyond the focus on these theoretical contributions, we conduct numerical examples in Section 5 to illustrate that the model has the potential to account jointly for the magnitudes of firm and unemployment dynamics.

2 Related Literature

Most work on labor search relies on undirected search with bargaining, where the number of matches is determined as a function of the ratio of unemployed workers to vacant jobs. Directed search (or competitive search) retains the idea of the matching function, but different contracts form different markets and workers can decide where to search. In most of these models, each firm can hire at most one worker in any given period. Properties of the matching function are reviewed in Petrongolo and Pissarides (2001), and Mortensen and Pissarides (1999) and Rogerson, Shimer, and Wright (2005) review models of undirected and directed search.

Closely related to our contribution is the work by Garibaldi and Moen (2010). They consider a directed search model where workers can search on-the-job and firms can hire many workers. They derive a number of new insights for on-the-job search. In contrast to our paper, though, they consider constant returns production functions, and the only determinant of firm size arises from convex vacancy creation costs. Even in the standard search model a linear frontier eliminates inefficiencies as long as the bargaining power is set according to the Hosios (1990) condition, irrespective of convex vacancy creation costs, since employment of any one worker does not diminish the marginal product of another worker.[5] In contrast, the difference between average

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[5]By the standard model we mean the model with random search and bargaining when only the

and marginal product in large firms and the possible complications for wage setting and efficiency is exactly what is at the heart of this study. Moreover, due to constant returns the current size of the firm ceases to be a state variable in Garibaldi and Moen (2010), and therefore firm growth and wage levels depend only on its productivity type but are independent of firm size. That is, the model is silent about the role of employer size and age for job creation. Additionally, it might be non-obvious that it is the convexity of the vacancy posting cost that limits the immediate jump in employment to the desired size. We additionally consider the possibility that screening and training of prospective employees requires time involvement of existing employees, which might be the more relevant restriction in the expansion of a firm. Firm–specific knowledge needs to be transmitted to the prospective hires, and this naturally limits the growth of the firm. Finally, our analysis also allows idiosyncratic and aggregate shocks.[6]

For settings where each firm can only hire one worker, the combination of wage competition and matching frictions that underlies directed search has been micro-founded as a coordination game in Peters (1991), Burdett, Shi, and Wright (2001) and Galenianos and Kircher (2009). In such games, sometimes multiple workers apply for the same job and only one of them can be hired. These micro–foundations can be extended to our multi–worker firms if one assumes that excess applicants for one position cannot fill another position at the same firm. This arises if different vacancies relate to different qualifications: for example, vacancies for an electrician cannot be filled by applicants for the position of a mechanic or a carpenter, even though each position yields roughly the same contributions in terms of marginal product (therefore labor productivity is modeled homogeneously). An alternative interpretation is that workers are literally identical and excess capacity can be substituted from one job to another, which means that posting additional jobs exhibits increasing returns; see Burdett, Shi, and Wright (2001), Hawkins (2006) and Lester (2009) for variations along these lines. In accordance with most work on large firms,

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unemployed can search, where the efficiency result for constant returns was shown e.g. in Cahuc and Wasmer (2001). We are not aware of efficiency results when on-the-job search is introduced into such a model.

[6]Productivity shocks might induce some firms to shed some of their workforce. We note that a linear frontier such as in Garibaldi and Moen (2010) would imply that a firm that fires some workers will fire all of them, unless there are strictly convex firing costs.

we adopt the first interpretation and abstract from possible increasing returns in hiring.

Hawkins's (2006) contribution considers a directed search model where firms hire multiple workers and produce output according to a decreasing–returns production function. He assumes that firms post a *finite* number of vacancies, and then receive a stochastic number of applications. Since the number of applicants is stochastic, he shows that posting a wage alone is not sufficient to induce efficiency. Rather, the posted contract has to condition on the realized number of applicants. These contingencies make the model somewhat intractable, and results on efficiency and firm dynamics out of steady state are missing. The standard assumption for large firms that many vacancies are posted eliminates some of the random elements. In fact, we assume that any firm advertises a *continuum* of vacancies (and employs a continuum of workers), thus eliminating uncertainty about the number of applicants entirely. By implication, the contract does not have to condition on the number of applicants, which can be inferred in advance. This feature simplifies the model considerably and allows us to obtain explicit and tractable solutions for firm dynamics.

Gourio and Rudanko (2009) consider a model of the product market where firms acquire a customer base. Some of the elements of their model are related, in the sense that a firm tries to attract a continuum of customers through price offers, and is limited by the current sales force. The research question is very different, though, in that their model does not speak to unemployment questions, and their analysis does not consider efficiency. Rather, they consider the relation between Tobin's q and investment.

This work focuses primarily on job flows, and all worker flows are transitions between unemployment and employment. Work following the lines of Burdett and Mortensen (1998), Moscarini and Postel-Vinay (2009b), Menzio and Shi (2008), and Garibaldi and Moen (2010) focussed additionally on worker flows between firms, and used this feature to address some puzzles in the literature. Except for the last contribution, none of these models allows a given firm to increase the number of vacancies that it posts, and so employment growth has to arise solely out of additional hiring per vacancy. None of these contributions allow for a difference between average and marginal productivity of labor.[7] A natural next step seems to be the merger of on-

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[7]Mortensen (2009) develops a model with on-the-job-search, decreasing-returns production func-

the-job search as in Garibaldi and Moen (2010) with a notion of decreasing returns on the intensive margin as explored here.

In addition to the relationships to random search models mentioned in the introduction, it is worth pointing out that current applied work on business cycles using multi-worker random search only focusses on the intensive margin of hiring by considering a fixed number of firms (Elsby and Michaels (2008), Fujita and Nakajima (2009)). Similarly, Cooper, Haltiwanger, and Willis (2007) assess business cycle implications for a fixed number of firms assuming that firms make each period take-it-or-leave offers without commitment to future wages, which is equivalent to bargaining with zero bargaining power for the workers. Our paper addresses additionally the entry and exit of firms, and this feature is in fact decisive to obtain a tractable solution based on block recursivity. The problem that bargaining might introduce unwarranted inefficiencies by assumption has also spurred other solutions than ours. For example, Veracierto (2008) and Samaniego (2008) consider general–equilibrium versions of the Hopenhayn and Rogerson (1993) model with frictionless labor markets and competitive wage setting. These approaches eliminate involuntary unemployment altogether. Veracierto (2009) introduces unemployment in an adaptation of the Lucas-Prescott island model that includes recruitment technologies. Competitive search allows the market to operate through decentralized wage setting, which attracts workers that optimally choose between search markets and are matched according to a standard matching function.

3 A stationary model of firm creation and firm growth

3.1 The environment

The model is set in discrete time and is stationary; that is, there are neither idiosyncratic nor aggregate shocks.

Workers and firms

tions, random search and Stole–Zwiebel wage bargaining.

There is a continuum of workers and firms, and workers are negligibly small relative to firms. That is, every active firm employs a continuum of workers.[8] The mass of workers is normalized to one. Each worker is infinitely–lived, risk–neutral, and discounts future income with factor $\beta < 1$. A worker supplies one unit of labor per period and receives income $b \geq 0$ when unemployed. On the other side of the labor market is an endogenous mass of firms. Firms are also risk neutral and have the same discount factor β. Upon entry, the firm pays a set–up cost $K > 0$ and draws productivity x with probability $\pi_0(x)$ from the finite set $x \in X$. In this section, productivity stays constant during the life of the firm. In each period, a firm produces output $xF(L)$ with $L \geq 0$ workers, where F is a differentiable, strictly increasing and strictly concave function satisfying $F'(\infty) = 0$. Firms die with exogenous probability $\delta > 0$ in which case all its workers are laid off into unemployment. Furthermore, each employed worker quits the job with exogenous probability $s \geq 0$. Thus, workers' separation probability is exogenous at $\eta \equiv 1 - (1 - \delta)(1 - s)$.

Recruitment

Search for new hires is a costly activity. A firm with current workforce L that posts V vacancies incurs recruitment cost $C(V, L, x)$ where C is differentiable and satisfies $C_1' > 0$, $C_1'' > 0$, $C_2' \leq 0$, $C_{12}'' \leq 0$, $C_{13}'' \geq 0$. We also assume that output net of recruitment cost $xF(L) - C(V, L, x)$ is strictly increasing in x for any (V, L, x). This formulation trivially covers the benchmark case where recruitment costs are independent of (L, x) and strictly convex in V (for applications of this specification, see e.g. Cooper, Haltiwanger, and Willis (2007), Koeniger and Prat (2007), Garibaldi and Moen (2010)). We also use this simplification in the next section. The more general formulation has the main advantage that it captures more intuitive justifications for the *strict* convexity of the recruitment costs. Most notably, our specification allows for the possibility that each vacancy requires h units of labor input to screen and train prospective applicants. Even if pecuniary vacancy costs are linear, the overall costs are strictly convex since the workers involved in

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[8]Although the set of individuals has the same cardinality as the set of firms, it is helpful to think of the set of firms as a closed interval in $\mathbb{R}$, and the set of workers as a two–dimensional subset of $\mathbb{R}^2$. When both sets are endowed with the Lebesgue measure, an active firm employs a continuum of workers, albeit of mass zero.

hiring cannot participate in this period's production. In this case $C(V, L, x) = xF(L) - xF(L - hV) + cV$, where the difference in the first two terms captures the loss in output and the last term captures linear pecuniary costs of vacancy postings.[9] Concavity of F implies that C is strictly convex in V.

Search and matching

A recruiting firm announces a flat flow wage income w to be paid to its new hires for the duration of employment relation. The assumption that the firm offers the same wage to all its new hires is no restriction. Indeed, it is straightforward to show that it is not optimal to post vacancies with different wages at a given point in time.[10] Further, because of risk neutrality, it is no restriction to consider flat wage contracts, although this leads to the dispersion of flow wages within the firm, as we see below.

There is no search on the job. Unemployed workers observe all vacancy postings and direct their search towards wages promising the highest expected income value. As is standard in models of competitive search, the labor market segments into a continuum of submarkets identified by different wages. In any of these submarkets, unemployed workers and vacant jobs are matched according to a constant–returns matching technology. When λ is the unemployment–vacancy ratio in such a submarket, a vacancy is matched with a worker with probability $m(\lambda)$ and a worker finds a job with probability $m(\lambda)/\lambda$. The function m is differentiable, strictly increasing, strictly concave, and it satisfies $m(0) = 0$ and $m(\lambda) \leq \min(1, \lambda)$ for all $\lambda \in [0, \infty)$. The law of large numbers together with the assumption that workers are small relative to firms ensures that firms know with certainty that they hire $m(\lambda)V$ workers when they post V vacancies in some submarket with worker–job ratio λ.

Timing

Every period is divided into four stages. *First*, new firms are created and draw their

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[9]Clearly no more workers can be engaged in hiring than are present at the firm already. To get the hiring process started, we therefore need to assume that a newborn firm is endowed with some initial workforce L_e (e.g., the entrepreneurs) who can undertake the initial hiring or production. In this case the production function is defined on the interval $L \in [-L_e, \infty)$; and recruitment activities of any firm are then constrained by its labor endowment: $hV \leq L + L_e$.

[10]An intuition for this result is provided in the discussion of equation (8) below.

productivity. *Second*, production and search activities take place. *Third*, vacancies and unemployed workers are matched, and a fraction s of workers leave their firm. And *fourth*, a share δ of firms dies. Newly hired workers may never work (and receive no wage income) in the unlucky event that their employer exits the market at the end of the period.

3.2 Equilibrium

Given that there are no aggregate shocks, we characterize a stationary equilibrium where a constant number of firms enters the market in every period and where the workers' unemployment utility and reservation wage are constant over time.

Workers' search problem

Let $W(w)$ be the present value of income for a worker employed at wage w, and let U be the present value of income for worker who remains unemployed this period and searches optimally thereafter. The surplus of having a job with wage w over staying unemployed can be written as

$$W(w) - U = \frac{w - R}{1 - \beta(1 - \eta)} \ , \tag{1}$$

where $R = (1-\beta)U$ is the worker's reservation wage (flow utility of unemployment). Unemployed workers anticipate that different wages attract different worker–job ratios λ and thus differ in their job–finding probabilities $m(\lambda)/\lambda$. They choose between all combinations $(w, \lambda) \in \Omega$ where Ω is the set of existing submarkets. Hence, they choose the submarket that gives the highest probability-weighted surplus. Weighting the surplus in (1) by the job finding probability $m(\lambda)/\lambda$ and accounting for the probability that the employing firm remains active $1-\delta$, yields the expected surplus ρ from optimal job search:

$$\rho \equiv \max_{(w,\lambda)\in\Omega} \frac{m(\lambda)}{\lambda}(1 - \delta)\frac{w - R}{1 - \beta(1 - \eta)} \ . \tag{2}$$

Since any submarket can only have a strictly positive λ if indeed workers choose to search in that market, it has to be the case that under optimal job search $\frac{m(\lambda)}{\lambda}(1 - \delta)[W(w) - U] = \rho$, or equivalently

$$w = R + \frac{\lambda}{m(\lambda)}\frac{1 - \beta(1 - \eta)}{1 - \delta}\rho \quad \text{whenever} \quad \lambda > 0 \ . \tag{3}$$

This condition says that a firm can only recruit workers when its wage offer matches the workers' reservation wage plus a premium which is needed to attract workers into a submarket with job–finding probability $m(\lambda)/\lambda$. This premium is increasing in λ. The determination of worker-job ratios as the outcome of optimal search decisions by workers is standard in the competitive search literature (e.g., Moen (1997), Acemoglu and Shimer (1999b)).[11] Finally, the reservation wage is linked to ρ from the unemployed worker's Bellman equation:

$$R \equiv (1 - \beta)U = b + \beta\rho . \tag{4}$$

Firms' recruitment policy

Let $J^x(L, W)$ be the profit value of a firm with productivity x, an employment stock of L workers and a commitment to a total wage bill of W. An entrant firm's profit value is then $J^x(0, 0)$.

The firm's recruitment choice involves deciding the number of posted vacancies V as well as the submarket where these vacancies are posted, characterized by the tuple (w, λ). Its recursive profit maximization problem is expressed as

$$
\begin{aligned}
J^x(L, W) \;=\; & \max_{(w,\lambda,V)} xF(L) - W - C(V, L, x) + \beta(1 - \delta)J^x(\hat{L}, \hat{W}) , \tag{5}\\
& \text{s.t. } \hat{L} = L(1 - s) + m(\lambda)V , \; \hat{W} = W(1 - s) + m(\lambda)Vw , \\
& V \geq 0 , \text{ and condition (3) .}
\end{aligned}
$$

Because of $m(0) = 0$, we can substitute (3) into the dynamic equation for the firm's wage bill to obtain

$$\hat{W} = (1 - s)W + V\left[\lambda\rho\frac{1 - \beta(1 - \eta)}{1 - \delta} + m(\lambda)R\right] .$$

The observation that the firm's objective and constraints are separable in L and W and linear in W suggests that the value function takes the form $J^x(L, W) = -BW + G^x(L)$ for some constant $B > 0$. Using this conjecture, the first–order

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[11]Note that we view each vacancy as a separate job requiring individual skills. See the discussion on increasing returns to hiring in Section 2.

conditions with respect to V and λ are

$$
\begin{aligned}
C_1'(V, L, x) &\geq \beta(1-\delta)\left\{m(\lambda)G^{x'}(\hat{L}) - B\left[\lambda\rho\frac{1-\beta(1-\eta)}{1-\delta} + m(\lambda)R\right]\right\}, \\
&\quad V \geq 0, \tag{6} \\
0 &\geq m'(\lambda)VG^{x'}(\hat{L}) - \left[\rho\frac{1-\beta(1-\eta)}{1-\delta} + m'(\lambda)R\right]VB, \quad \lambda \geq 0, \tag{7}
\end{aligned}
$$

which are both satisfied with complementary slackness. It is no loss of generality to consider only solutions where the first inequality in (7) binds,[12] so this equation can be substituted into (6) to get

$$
C_1'(V, L, x) \geq \beta\rho\frac{m(\lambda) - \lambda m'(\lambda)}{m'(\lambda)}, \quad V \geq 0, \tag{8}
$$

with complementary slackness. This condition describes intratemporal optimality between the two recruitment tools of the firm: the number of posted vacancies V on the one hand, and the posted wage w (and thus the worker–job ratio λ) on the other. The RHS is the firm's marginal revenue from an additional vacancy which is increasing in λ. The LHS is marginal cost of an additional vacancy; from the assumptions on C follows that it is (weakly) decreasing in L and strictly increasing in V. Hence the implicit solution to this equation is a function $V = V^x(L, \lambda)$ which is increasing in λ and (weakly) increasing in L. Intuitively, with higher λ the probability to fill a vacancy increases, and hence the firm is willing to bear higher marginal recruitment costs by advertising more jobs. And with higher L, marginal recruitment costs fall and hence the firm is inclined to post more vacancies. Equation (8) also provides an intuition why it is never optimal to offer different wages at a given point in time: any one of its V vacancies adds marginal cost C_1'; this marginal cost must be balanced against the marginal return of the vacancy which depends upon the job–filling rate; Hence, the firm will choose the same λ (and thus post the same wage) for any of its vacancies. Moreover, since the RHS attains any value between 0 and $+\infty$ as λ varies between 0 and $+\infty$, the firm will always recruit workers when λ is large enough and it will never recruit at low enough values of λ (unless $C_1'(0, L, x) = 0$).

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[12]It follows from the complementary–slackness condition (6) that $\lambda = 0$ implies $V = 0$. This "no recruitment" solution is also obtained from (8).

The envelope conditions for problem (5) are

$$G^{x'}(L) = xF'(L) - C_2'(V, L, x) + \beta(1 - \eta)G^{x'}(\hat{L}) , \qquad (9)$$

$$B = 1 + \beta(1 - \eta)B . \qquad (10)$$

This shows that $B = 1/[1 - \beta(1 - \eta)]$ is a constant. Condition (9) together with the first–order conditions (6) and (7) further confirms that the conjecture $J^x(L, W) = -BW + G^x(L)$ is correct: the firm's choices of λ and V depend on the employment stock L but not on the wage bill W. The firm's value and policy functions can be characterized as follows.

Proposition 1: *For a given value $\rho > 0$ and the corresponding reservation wage R defined by (4), the firm's value function is $-BW + G^x(L; \rho)$ where G^x is strictly increasing, strictly concave and differentiable in L, continuous and decreasing in ρ, and increasing in x. The firm's policy function $\lambda^x(L)$ is strictly decreasing in L and strictly increasing in x whenever $\lambda^x(.) > 0$. Posted vacancies $V^x(L, \lambda)$ are weakly increasing in L and strictly increasing in λ whenever $V^x(L, \lambda) > 0$.*

Proof: Appendix.

Corollary: *Conditional on productivity, younger (and smaller) firms pay higher wages and have a higher job–filling rate. Conditional on firm size, more productive firms pay higher wages and have a higher job–filling rate.*

To see how the firm grows over time, consider a firm with productivity x that enters in some period τ. Its job creation policy is then described by a sequence $(L_t, \lambda_t, V_t)_{t \geq \tau}$ starting from $L_\tau = 0$. Posted vacancies $V_t = V^x(L_t, \lambda_t)$ are the implicit solution of equation (8). The employment stock accumulates according to

$$L_{t+1} = (1 - s)L_t + m(\lambda_t)V^x(L_t, \lambda_t) . \qquad (11)$$

And from (7) and (9) follows the Euler equation

$$xF'(L_{t+1}) - C_2'(V_{t+1}, L_{t+1}, x) - R = \frac{\rho}{1 - \delta}\left[\frac{1}{m'(\lambda_t)} - \frac{\beta(1 - \eta)}{m'(\lambda_{t+1})}\right] . \qquad (12)$$

This equation says that the firm recruits fast in period t (i.e. λ_t is large) when the expected marginal product in $t + 1$ is high. In the example with recruitment cost

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$C(V, L, x) = xF(L) - xF(L - hV) + cV$, equations (12) and (8) can be further simplified to an equation which is independent of L_t:[13]

$$\beta\rho\Big[m(\lambda_{t+1}) - \lambda_{t+1}m'(\lambda_{t+1})\Big] - [Rh + c]m'(\lambda_{t+1}) = \frac{\rho h}{1 - \delta}\left[\frac{m'(\lambda_{t+1})}{m'(\lambda_t)} - \beta(1 - \eta)\right]. \quad (13)$$

In Lemma 1 of the Appendix, we show that this equation has a unique steady state $\lambda^* > 0$ if recruitment costs are low enough, and λ_t converges to λ^* from any initial value $\lambda_\tau > 0$. Figure 1 shows the phase diagram for the system (11) and (13). The curve where the employment stock is constant is downward sloping since (8) implies that $V^x(L, \lambda)/L$ is increasing in L. If the condition

$$xF'(0) > R + \frac{\rho[1 - \beta(1 - \eta)]}{(1 - \delta)m'(\lambda^*)}$$

holds, there exists a unique stationary employment level $L^* > 0$. The corresponding dynamics imply further that there is a downward–sloping saddle path converging to the long–run employment level. Graphically, the firm's policy function $\lambda^x(L)$ traces this saddle path.

It follows from these considerations that the firm's recruitment policy is characterized by a path of declining wage offers and job–filling rates along the transition to the firm's long–run employment level. Concavity of the firm's production function implies that the firm wants to spread out its recruitment costs across several periods. This statement remains true for other forms of the recruitment technology. Only when recruitment costs are linear in vacancies, $C(V) = cV$, the firm would choose a constant λ^* (and hence post the same wage in all periods). In that case, it would recruit L^* workers in the entry period and then keep the employment level constant. As soon as recruitment costs are strictly convex, such a policy is not optimal, and it may not be feasible due to the capacity constraint on labor input in recruitment. This feature of our model is similar to models of optimal investment with convex adjustment costs. A novel feature is that the firm's wage policy reflects the desire to recruit fast in the start–up phase and more slowly in the convergence phase. This finding is in line with the empirical observation that wages are higher in fast–growing firms. On the other hand, the model's predictions on the relation between productivity, firm size and wages are standard; more productive firms pay

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[13]This equation becomes an inequality in the no–recruitment case $V^x(L_{t+1}, \lambda_{t+1}) = 0$.
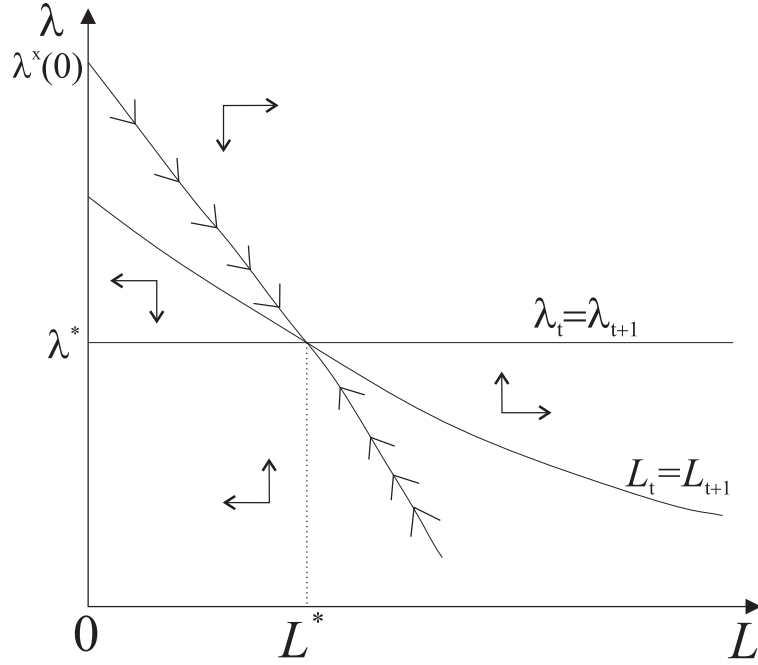
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Figure 1: The firm's optimal recruitment policy follows the declining saddle path.

higher wages, they are bigger, and they also grow faster in the start–up phase. Furthermore, there is within–firm wage dispersion; workers receive higher wages when they were hired in an earlier phase of the firm's life.[14]

Firm creation

No entrant makes a positive profit when the expected profit income of a new firm equals the entry cost, that is,

$$\sum_{x \in X} \pi(x) J^x(0,0) = K \ . \tag{14}$$

This condition implicitly pins down the worker's job surplus ρ and therefore, via the firm's optimal recruitment policy, worker–job ratios in all submarkets. In a

---

[14]Because of risk–neutrality, there is a payoff–equivalent equilibrium where each firm pays the same constant flow wage income to all its workers in addition to an initial hiring premium which is declining as the firm is growing.

stationary equilibrium, a constant mass of N_0 firms enters the market in every period, so that there are $N_a = N_0(1 - \delta)^a$ firms of age a in any period. Let $(L_a^x, \lambda_a^x, V_a^x)_{a \geq 0}$ be the employment/recruitment path for a firm with productivity x. Then, a firm of age a with productivity x has L_a^x employed workers, and $\lambda_a^x V_a^x$ unemployed workers are searching for jobs in the same submarket where this firm searches for workers. Because there is a unit mass of workers, the mass of entrant firms N_0 is uniquely pinned down from aggregate resource feasibility:

$$1 = \sum_{a \geq 0} N_0(1 - \delta)^a \sum_{x \in X} \pi(x)[L_a^x + \lambda_a^x V_a^x] . \tag{15}$$

General equilibrium

We now define a stationary equilibrium with positive firm creation. When K is large enough, there may also be an uninteresting equilibrium without firms which is ignored in the following.

Definition: A *stationary competitive search equilibrium* is a list

$$\left(\rho, R, N_0, (L_a^x, \lambda_a^x, V_a^x)_{x \in X, a \geq 0} \right)$$

such that

(a) Unemployed workers' job search strategies maximize utility. That is, the reservation wage R satisfies (4) with ρ defined in (2).

(b) Firms' recruitment policies are optimal. That is, given ρ and R, and for all $x \in X$, $(L_a^x, \lambda_a^x, V_a^x)_{a \geq 0}$ describes the firm's growth path, obtained from the policy functions solving problem (5).

(c) There is free entry of firms, equation (14).

(c) The number of entrant firms is consistent with aggregate resource feasibility, equation (15).

Proposition 2: *A stationary competitive search equilibrium with active firms exists and is unique, provided that K is sufficiently small and $F'(0)$ is sufficiently large.*

There is wage dispersion both within and across firms. Wages are increasing in firm productivity and decreasing in firm age. Conditional on firm age, larger firms pay higher wages.

Proof: Appendix.

3.3 Efficiency

The social planner decides at each point in time about firm creation, vacancy creation and worker–job ratios in different submarkets of the economy. The planner takes as given the numbers of firms that were created in some earlier period, as well as the employment stocks of all these firms. Formally, the planner's state vector is

$$\sigma = (N_a, L_a^x)_{a \geq 1, x \in X} \ ,$$

where N_a is the mass of firms of age $a \geq 1$, and L_a^x is employment of a firm with productivity x and age a. Of course, it is no restriction to impose that all firms of a given type (a, x) are equally large. The planner maximizes the present value of output net of opportunity costs of employment and net of the costs of firm creation and vacancy creation. With $\hat{\sigma}$ to denote the state vector in the next period, the recursive formulation of the social planning problem is

$$
\begin{aligned}
S(\sigma) \ = \ & \max_{N_0, (V_a^x, \lambda_a^x)_{a \geq 0}} \left\{ \sum_{a \geq 0} N_a \sum_{x \in X} \pi(x) \Big[xF(L_a^x)) - bL_a^x - C(V_a^x, L_a^x, x) \Big] \right\} - KN_0 + \beta S(\hat{\sigma}) \\
& \text{s.t. } L_0^x = 0, \ \hat{L}_{a+1}^x = (1-s)L_a^x + m(\lambda_a^x)V_a^x, \ a \geq 0, \ x \in X \ , \qquad (16) \\
& \hat{N}_{a+1} = (1-\delta)N_a \ , \ a \geq 0 \ , \\
& \sum_{a \geq 0} N_a \sum_{x \in X} \pi(x) \Big(L_a^x + \lambda_a^x V_a^x \Big) \leq 1 \ .
\end{aligned}
$$

The last condition is the economy's resource constraint. It states that the mass of of all individuals that are attached to some firm of type (a, x), either as workers L_a^x or as unemployed workers queuing up for a job at this firm $\lambda_a^x V_a^x$, may not exceed one. We say that a solution to problem (16) is *socially optimal.*

Proposition 3: *The stationary competitive search equilibrium is socially optimal.*

Proof: Appendix.

4 Productivity shocks and firm dynamics

We now extend the previous model to include both idiosyncratic (firm–specific) and aggregate productivity shocks. This extension allows us to explore not only two margins of job creation (firm entry and firm growth), but also the two margins of job destruction (firm exit and firm contraction). We simplify by assuming that recruitment costs $C(V)$ depend on vacancies alone; again, C is an increasing and convex function. Output of a firm with L workers is $xzF(L)$ where $x \in X$ is idiosyncratic productivity and $z \in Z$ is aggregate productivity. Both x and z follow Markov processes on finite state spaces X and Z with respective transition probabilities $\pi(x_+|x)$ and $\psi(z_+|z)$. An entrant firm pays fixed cost K and draws an initial productivity level $x \in X$ with probability $\pi_0(x)$. For a firm of age $a \geq 0$, let $x^a = (x_a, \ldots, x_0) \in X^{a+1}$ denote the history of idiosyncratic productivity, and let $z^t = (z_t, \ldots, z_0)$ be the history of aggregate shocks at time t. Write $\psi(z^t)$ and $\pi(x^a)$ for the unconditional probabilities of aggregate and idiosyncratic productivity histories.

We assume that an active firm incurs a fixed operating cost $f \geq 0$ per period. This parameter is required to obtain a non–trivial exit margin.[15] Each firm exits with exogenous probability $\delta_0 \geq 0$ which is a lower bound for the actual exit rate $\delta \geq \delta_0$. Similarly, workers quit a job with exogenous rate $s_0 \geq 0$ which provides a lower bound for the actual separation rate $s \geq s_0$.[16]

The timing within each period is as follows. *First*, aggregate productivity is revealed, new firms enter, all existing firms draw their idiosyncratic productivities and decide about exit. *Second*, firms decide about recruitment and separations, and recruiting firms are matched with unemployed workers. An unemployed worker who has just left another job (due to firm exit, quit or layoff) can search for reemployment within the same period. And *third*, production takes place. In the following, we first describe the planning problem before we show its equivalence to a competitive–search equilibrium in Section 4.4.

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[15] A non–trivial exit margin would also obtain in the presence of firing costs when $f = 0$.

[16] Although this model ignores many important worker flows, such as those between jobs and the flows in and out of the labor force, s_0 represents a measure of exogenous worker turnover, as in Fujita and Nakajima (2009).

4.1 The planning problem

The planner decides at each point in time about firm entry and exit, layoffs and job creation, as well as worker–job ratios in different submarkets. In a given aggregate history z^t, we denote by $N(x^a, z^t)$ the mass of firms of age a with idiosyncratic history x^a. Similarly, $L(x^a, z^t)$ is the employment stock of any of these firms. At every history node z^t and for every firm type x^a, the planner decides an exit probability $\delta(x^a, z^t) \geq \delta_0$, a separation rate $s(x^a, z^t) \geq s_0$, vacancy postings $V(x^a, z^t) \geq 0$, and a worker–job ratio $\lambda(x^a, z^t)$ for the submarket in which vacancies of that firm are matched with unemployed workers.[17] The numbers of firm types change between periods $t-1$ and t according to

$$N(x^a, z^t) = [1 - \delta(x^a, z^t)]\pi(x_a|x_{a-1})N(x^{a-1}, z^{t-1}) \,, \tag{17}$$

and the employment stock at any of these firms adjusts to

$$L(x^a, z^t) = [1 - s(x^a, z^t)]L(x^{a-1}, z^{t-1}) + m(\lambda(x^a, z^t))V(x^a, z^t) \,. \tag{18}$$

At time $t = 0$, the planner takes as given the numbers of firms that entered the economy in some earlier period, as well as the employment stock of each of these firms. Hence, the state vector at date 0, prior to the realization of productivities, is summarized by the initial firm distribution $(N(x^{a-1}, .), L(x^{a-1}, .))_{a \geq 1, x^{a-1} \in X^a}$. In a given history z^t, the planner also decides the mass of new entrants $N_0(z^t) \geq 0$, so that

$$N(x_0, z^t) = [1 - \delta(x_0, z^t)]\pi_0(x_0)N_0(z^t) \text{ and } L(x_0, z^t) = m(\lambda(x_0, z^t))V(x_0, z^t) \,. \tag{19}$$

The planning problem is

$$\max_{\delta, s, V, \lambda, N_0} \sum_{t \geq 0, z^t} \beta^t \psi(z^t) \Bigg\{ - KN_0(z^t) + \tag{20}$$

$$\sum_{a \geq 0, x^a} N(x^a, z^t)\Big[x_a z_t F(L(x^a, z^t)) - bL(x^a, z^t) - f - C(V(x^a, z^t))\Big] \Bigg\}$$

---

[17]To save on notation, we do not allow the planner to discriminate between workers with different firm tenure. Given that there is no learning-on-the-job, there is clearly no reason for the planner to do so. Nonetheless, the competitive search equilibrium considered in 4.4 allows firms to treat workers in different cohorts differently, which is necessary because each firm offers contracts sequentially and is committed to these contracts. See the proof of Proposition 6 for further discussion of that issue.

subject to the dynamic equations for N and L, namely (17), (18) and (19), and subject to the resource constraints, for all $z^t \in Z^{t+1}$,

$$\sum_{a \geq 0, x^a} N(x^a, z^t) \Big[(1 - s(x^a, z^t)) L(x^{a-1}, z^{t-1}) + \lambda(x^a, z^t) V(x^a, z^t) \Big] \leq 1 . \qquad (21)$$

This constraint says that the labor force (employment plus unemployment) cannot exceed the given unit mass of workers. The first part of the sum, namely

$$\sum_{a \geq 0, x^a} N(x^a, z^t)(1 - s(x^a, z^t)) L(x^{a-1}, z^{t-1}) ,$$

are the workers that are employed in some firm after separations have taken place. The remaining part of the sum are unemployed workers queueing up for employment in one of the active firms posting $V(.)$ vacancies in submarkets with worker–job ratios $\lambda(.)$. For instance, there are $N(x^a, z^t)$ active firms with productivity history x^a, each of which posts $V(x^a, z^t)$ vacancies that meet $\lambda(x^a, z^t) V(x^a, z^t)$ unemployed workers in a particular submarket.

4.2   Characterization of the planning solutions

There is a convenient characterization of a planning solution which says that exit, layoff, and hiring decisions follow a recursive equation at the level of the individual firm. Let $\beta^t \psi(z^t) \mu(z^t) \geq 0$ be the multiplier on the resource constraint (21) in history z^t. Intuitively, $\mu(z^t)$ is the social value of a worker in history z^t. Let $G_t(L, x, z^t)$ denote the social value of an existing firm with employment stock L, idiosyncratic productivity x and aggregate productivity history z^t. The sequence G_t obeys the recursive equations

$$
\begin{aligned}
G_t(L, x, z^t) \;=\; & \max_{\delta, s, V, \lambda} (1 - \delta) \Big\{ x z_t F(\hat{L}) - b\hat{L} - f - C(V) - \mu(z^t)[(1 - s)L + \lambda V] \\
& + \beta \sum_{\hat{x} \in X} \sum_{z_{t+1} \in Z} \pi(\hat{x}|x) \psi(z_{t+1}|z_t) G_{t+1}(\hat{L}, \hat{x}, (z_{t+1}, z^t)) \Big\}
\end{aligned}
\qquad (22)
$$

$$\text{s.t.} \quad \hat{L} = (1 - s)L + m(\lambda)V ,$$
$$\delta \in [\delta_0, 1], \;\; s \in [s_0, 1], \;\; \lambda \geq 0, \;\; V \geq 0 .$$

The interpretation of these equations is rather straightforward. The planner wants a firm with characteristics (L, x) to stay active in aggregate history z^t whenever the term in braces is non–negative, otherwise he sets $\delta = 1$. The term in braces gives the value of an active firm. In the current period, this value encompasses the firm's output net of the opportunity cost of employment, net of fixed costs and recruitment costs, and net of the social cost of workers tied to the firm in this period; these workers include those that are retained from the previous period, namely $(1 - s)L$, and also λV unemployed workers who aim to find a job at the firm (of which $m(\lambda)V \leq \lambda V$ eventually find a job).

Proposition 4: *There exist value functions $G_t : \mathbb{R}_+ \times X \times Z^{t+1} \to \mathbb{R}$, $t \geq 0$, satisfying the system of recursive equations (22). The firm dynamics obtained from the solution of the planning problem (20) is the same as the one obtained from (22), and optimal entry satisfies the complementary–slackness condition*

$$\sum_{x \in X} \pi_0(x) G_t(0, x, z^t) \leq K \ , \ N_0(z^t) \geq 0 \ . \tag{23}$$

Whilst Proposition 4 is a convenient characterization of planning solutions, it cannot be applied for computational purposes. The difficulty is that the multipliers $\mu(z^t)$ are non–stationary and depend on the initial firm distribution; furthermore, the firm–specific value functions G_t are defined on a high–dimensional state space. However, a much more powerful characterization can be obtained under the provision that firm entry is positive in all states of a particular planning solution, so that the first inequality in (23) is binding in all states of the world. When this is the case, the firm–level value function turns out to be independent of the firm distribution, a feature coined "block recursivity" by Menzio and Shi (2008). Crucially, the social value of a worker only depends on the current aggregate state and not on the state history.

To see this, suppose there are n aggregate states $z_1 \leq \ldots \leq z_n$ and let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}_+^n$ be a vector of social values in these states. Let $G^i(L, x, \mu)$ be the social value of a firm with employment stock L, idiosyncratic productivity x and aggregate productivity z_i, for $i = 1, \ldots, n$. $G = (G^i) : \mathbb{R}_+ \times X \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ satisfies

the Bellman equations

$$G^i(L, x, \mu) = \max(1 - \delta)\left\{ xz_i F(\hat{L}) - b\hat{L} - C(V) - \mu_i[(1 - s)L + \lambda V]\right. \quad (24)$$

$$\left. -f + \beta \sum_{\hat{x} \in X} \sum_{z_j \in Z} \pi(\hat{x}|x)\psi(z_j|z_i)G^j(\hat{L}, \hat{x}, \mu)\right\},$$

where maximization is subject to the same constraints as in problem (22). Positive entry in all aggregate states requires that the expected social value of a new firm is equal to the entry cost

$$\sum_{x \in X} \pi_0(x)G^i(0, x, \mu) = K \ . \quad (25)$$

This characterization of planning solutions by $(G^i, \mu_i)_{i=1,\ldots,n}$ is particularly helpful for numerical applications. Despite the high degree of heterogeneity, the model can be solved by a recursive problem on a low–dimensional state space (24) and the (simultaneous) solution of a finite–dimensional fixed point problem (25). Importantly, the distribution of firms is irrelevant for that computation. Ex post, the actual number of entrants, and hence the evolution of the firm distribution, is obtained as a residual in the economy's resource constraint in simulations of the model. Because the number of entrant firms $N_0(z^t)$ *does depend* on the full history of aggregate states (or, equivalently, on the current firm distribution), it cannot be proved mathematically that the planning solution has indeed positive entry in all state histories. Nonetheless, in any quantitative application of the model this possibility should not play a role. Further, the welfare theorem proven in the next section holds generally and hence does not rest upon positive entry in all states. Analytically, we can only prove that a planning solution with positive entry exists in the absence of aggregate shocks. For small aggregate shocks, we show that equations (24) and (25) have a solution.

Proposition 5:

(a) *If K, f, and b are sufficiently small and if $z_1 = \ldots = z_n = z$, equations (24) and (25) have a unique solution $G(L, x, \mu)$ with $\mu_1 = \ldots = \mu_n$. There exists a planning solution with positive entry and a stationary firm distribution.*

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(b) *If, moreover, the transition matrix $\psi(z_j|z_i)$ is strictly diagonally dominant and if $|z_i - z|$ is sufficiently small for all i, equations (24) and (25) have a unique solution.*

4.3 Recruitment and layoff strategies

The reduction of the planning solution to problem (22) permits a straightforward characterization of the optimal layoff and hiring strategies. A firm with productivity xz and employment stock L should dismiss workers (that is, $s > s_0$) in state $i = 1, \ldots, n$ iff

$$xz_i F'(L(1 - s_0)) - b - \mu_i + \beta \sum_{\hat{x}} \sum_{z_j} \pi(\hat{x}|x)\psi(z_j|z_i)G^{j'}(L(1 - s_0), \hat{x}, \mu) < 0 \ . \quad (26)$$

This expression is the marginal social surplus of a worker at the employment stock $L(1 - s_0)$ after worker turnover. If marginal worker surplus is negative, the firm lays off some workers so as to equate the left–hand side to zero.

Conversely, for the firm to recruit workers, it must be that $\lambda > 0$ and $V > 0$. In that case, it follows from the first–order conditions for λ and V that[18]

$$C'(V) = \mu_i \lambda \Big(\frac{m(\lambda)}{\lambda m'(\lambda)} - 1 \Big) \ . \quad (27)$$

As in the previous section, it follows from concavity of m and convexity of C that there is an increasing relation between the worker–job ratio and the number of posted vacancies at the firm. With higher λ, the probability to fill a vacancy increases, and hence the planner is willing to post more vacancies at higher marginal recruiting cost. Denote the solution to equation (27) by $V = V_i(\lambda)$. It is straightforward to see that $V_i(\lambda) \to \infty$ when $\lambda \to \infty$, and there is some $\underline{\lambda}_i \geq 0$ such that $V_i(\lambda) > 0$ iff $\lambda > \underline{\lambda}_i$. Note also that the function $V_i(\lambda)$ is independent of the firm's idiosyncratic productivity and of its current employment stock. The planner's optimal choice of λ for firm (L, x) in aggregate state i satisfies

$$xz_i F'(\hat{L}) - b + \beta \sum_{\hat{x}} \sum_{z_j} \pi(\hat{x}|x)\psi(z_j|z_i)G^{j'}(\hat{L}, \hat{x}, \mu) = \frac{\mu_i}{m'(\lambda)} \ ,$$

---

[18]This equation is analogous to (8). See the derivation of (47) in the proof of Proposition 3.

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with $\hat{L} = L(1 - s_0) + m(\lambda)V_i(\lambda)$. Therefore, the firm recruits workers, if and only if

$$xz_i F'(L(1 - s_0)) - b + \beta \sum_{\hat{x}} \sum_{z_j} \pi(\hat{x}|x)\psi(z_j|z_i)G^{j'}(L(1 - s_0), \hat{x}, \mu) > \frac{\mu_i}{m'(\underline{\lambda})} \ . \quad (28)$$

The two conditions (26) and (28) illustrate how the firm's strategy depends on its characteristics (L, x). Small and productive firms recruit workers and grow, whereas large and unproductive firms dismiss workers and shrink. Depending on the functional forms for $C(.)$ and $m(.)$, there can also be an open set of characteristics where firms do not adjust their workforce.[19]

4.4 Decentralization

We now describe a competitive search equilibrium that gives rise to the same allocation as the planning solution characterized in Proposition 4. Firms offer workers a sequence of state–contingent wages, to be paid for the duration of the match. They also commit to cohort–specific and state–contingent separation probabilities. Contracts are contingent on the idiosyncratic productivity history of the firm at age k, x^k, and on the aggregate state history z^t at time t. Formally, a contract offered by a firm of age a at time T takes the form

$$C_a = \left(w_a(x^k, z^{T+k-a}), \varphi_a(x^k, z^{T+k-a}) \right)_{k \geq a} ,$$

where $w_a(x^k, z^t)$ is the wage paid to the worker in firm history (x^k, z^t), conditional on the worker being still employed by the firm in that instant. $\varphi_a(x^k, z^t) \geq \delta_0 + (1 - \delta_0)s_0$, for $k > a$, is the probability of a job separation prior to the production stage in history x^k. In the hiring period, a separation cannot occur, so $\varphi_a(x^a, z^T) = 0$ by definition.

The workers' search problem

Let $U(z^t)$ be the utility value of an unemployed worker in history z^t, and let $W(C_a, x^k, z^t)$ be the utility value of a worker hired by a firm of age a in contract C_a

---

[19]Similar patterns for employment adjustment are obtained in the models of Bentolila and Bertola (1990) and Elsby and Michaels (2008).

who is currently employed at that firm in history x^k, with $k \geq a$. The latter satisfies the recursive equation

$$
\begin{aligned}
W(C_a, x^k, z^t) &= \varphi_a(x^k, z^t)U(z^t) + (1 - \varphi_a(x^k, z^t))\Big[w_a(x^k, z^t) \quad\quad (29) \\
&\quad + \beta \sum_{x_{k+1}} \sum_{z_{t+1}} \pi(x_{k+1}|x_k)\psi(z_{t+1}|z_t)W(C_a, x^{k+1}, z^{t+1})\Big] .
\end{aligned}
$$

An unemployed worker searches for contracts which promise the highest expected utility, considering that more attractive contracts are less likely to find. The worker observes all contracts C_a and he knows that the probability to sign a contract is $m(\lambda)/\lambda$ when λ is the worker–job ratio in the submarket where the contract is offered. That is, potential submarkets are parameterized by the tuple (λ, C_a). Unemployed workers enter those submarkets where expected surplus is maximized:

$$
\rho(z^t) = \max_{(\lambda, C_a)} \frac{m(\lambda)}{\lambda}\Big[W(C_a, x^a, z^t) - b - \beta E_{z_t}U(z^{t+1})\Big] . \quad\quad (30)
$$

Because an unemployed worker gets one chance to search in every period, his Bellman equation reads as

$$
U(z^t) = b + \rho(z^t) + \beta E_{z_t}U(z^{t+1}) . \quad\quad (31)
$$

The firms' problem

A firm of age a with history (x^a, z^t) takes as given the employment stocks of workers hired in some earlier period, $(L_\tau)_{\tau=0}^{a-1}$, as well as the contracts signed with these workers, $(C_\tau)_{\tau=0}^{a-1}$. The firm chooses an exit probability δ and cohort–specific layoff probabilities s_τ. For these probabilities to be consistent with separation probabilities specified in existing contracts, it must hold that $\delta \leq \varphi_\tau(x^a, z^t)$ for all $\tau \leq a - 1$, and $s_\tau = 1 - (1 - \varphi_\tau(x^a, z^t))/(1 - \delta)$ when $\delta < 1$, with arbitrary choice of s_τ when $\delta = 1$. The firm also decides new contracts C_a to be posted in V vacancies in a submarket with worker–job ratio λ. It is no restriction to presuppose that the firm offers only one type of contract and searches in only one submarket. When J_a is the value function of a firm of age a, the firm's problem is written as

$$
\begin{aligned}
J_a\Big[(C_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] &= \max_{(\delta, \lambda, V, C_a)} (1 - \delta)\Big\{ x_a z_t F\Big(\sum_{\tau=0}^{a} \hat{L}_\tau\Big) - W - f \quad (32) \\
&\quad - C(V) + \beta \sum_{x_{a+1}} \sum_{z_{t+1}} \pi(x_{a+1}|x_a)\psi(z_{t+1}|z_t)J_{a+1}\Big[(C_\tau)_{\tau=0}^{a}, (L_\tau)_{\tau=0}^{a}, x^{a+1}, z^{t+1}\Big]\Big\}
\end{aligned}
$$

s.t. $\quad \hat{L}_a = m(\lambda)V, \;\; , \lambda \geq 0, \; V \geq 0, \; \hat{L}_\tau = L_\tau \dfrac{1 - \varphi_\tau(x^a, z^t)}{1 - \delta}, \;\; \tau \leq a - 1 \;, \quad$ (33)

$$\delta \in [\delta_0, \min_{0 \leq \tau \leq a-1} \varphi_\tau(x^a, z^t)], \;\; s_0(1 - \delta) \leq (1 - \varphi_\tau(x^a, z^t)) \;, \tag{34}$$

$$W = \sum_{\tau=0}^{a} w_\tau(x^a, z^t) \hat{L}_\tau \;, \tag{35}$$

$$W(C_a, x^a, z^t) \geq b + \beta E_{z_t} U(z^{t+1}) + \frac{\lambda \rho(z^t)}{m(\lambda)} \text{ when } \lambda > 0 \;. \tag{36}$$

The last condition is the workers' participation constraint; it specifies the minimum expected utility that contract C_a must promise in order to attract a worker queue of length λ per vacancy. In (33), the ratio $(1 - \varphi_\tau(x^a, z^t))/(1 - \delta)$ is (arbitrarily) set to zero when $\delta = \varphi_\tau(x^a, z^t) = 1$.

Definition: A (stationary) competitive search equilibrium is a list

$$\left[U(z^t), \rho(z^t), C_a(x^a, z^t), \lambda(x^a, z^t), V(x^a, z^t), \delta(x^a, z^t), J_a(.), (L_\tau(x^a, z^t))_{0 \leq \tau \leq a}, N(x^a, z^t), N_0(z^t) \right],$$

for all $t \geq 0$, $a \geq 0$, $x^a \in X^{a+1}$, $z^t \in Z^{t+1}$, and for a given initial firm distribution, such that

(a) Firms' exit, hiring and layoff strategies are optimal. That is, J_a is the value function and $C_a(.)$, $\delta(.)$, $\lambda(.)$, and $V(.)$ are the policy functions for problem (32).

(b) Employment evolves according to

$$
\begin{aligned}
L_\tau(x^a, z^t) &= L_\tau(x^{a-1}, z^{t-1}) \frac{1 - \varphi_\tau(x^a, z^t)}{1 - \delta(x^a, z^t)} \;, \;\; 0 \leq \tau \leq a - 1 \;, \\
L_a(x^a, z^t) &= m(\lambda(x^a, z^t)) V(x^a, z^t) \;, \;\; a \geq 0 \;.
\end{aligned}
$$

(c) Firm creation is optimal. That is, the complementary slackness condition

$$\sum_x \pi_0(x) J_0(x, z^t) \leq K \;, \;\; N_0(z^t) \geq 0 \tag{37}$$

holds, and the number of firms evolves according to $N(x_0, z^t) = \pi_0(x_0)[1 - \delta(x_0, z^t)] N_0(z^t)$ and (17).

(d) Workers' search strategies are optimal, i.e. (ρ, U) satisfy equations (30) and (31).

(e) Aggregate resource feasibility:

$$\sum_{a \geq 0, x^a} N(x^a, z^t) \left[\lambda(x^a, z^t) V(x^a, z^t) + \sum_{\tau=0}^{a-1} L_\tau(x^a, z^t) \right] = 1 . \qquad (38)$$

Proposition 6: *The competitive search equilibrium is socially optimal.*

Proof: Appendix.

Discussion of wages and employment commitment

It is not hard to see that a wage commitment is sufficient for a firm to implement its desired policy, even if it cannot commit to separation rates. Given risk neutrality, the firm can set the wages following any future history exactly equal to his reservation wage which is the sum of unemployment income and the worker's shadow value $b + \mu(z^t)$. It can achieve any initial transfer to attract workers through an initial hiring bonus. The costs of an existing worker therefore always equal his value in his best available alternative: unemployment and search for another job. Since the flow surplus for any retained worker equals his shadow value, the firm's problem in this case coincides with the planner's problem (22), and firing will be exactly up to the socially optimal level even though the firm only commits to the wages and not to the employment levels. Workers do not have any incentive to quit the job unilaterally, either, because they are exactly compensated for their social shadow value from searching, which in this setting is equal to their personal shadow value.

Similarly, given employment commitment the wage–tenure profiles for individual workers are arbitrary because of risk–neutrality, as long as they satisfy the workers' participation constraint with equality. As we show in the proof of Proposition 6, firms do not need to discriminate in separation rates between workers in different cohorts. Nonetheless, such equilibria are also possible; then workers with higher separation rates will be compensated through higher wage transfers, whereas workers with more stable jobs earn lower wages. Put differently, this model does not say anything about individual wage–tenure profiles. It only pins down the surplus split between workers and firms.

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In our numerical examples, we consider the benchmark case where wage profiles are not dispersed within the firm. That is, all workers within firm (L, x) in history z^t earn the same flow wage $w(L, x, z^t)$. In a block–recursive equilibrium, such a wage profile can be easily calculated using (29) and the binding condition (36).

5   A numerical example

HIGHLY PRELIMINARY

Consider the following illustrative example with arbitrary parameter choices.

Consider first a stationary economy without aggregate shocks to generate a stationary firm distribution. Set $\beta = 0.95$ and suppose that idiosyncratic productivities are drawn from $X = \{1.2, 1.5, 1.8, 2.1, 2.4\}$. Upon entry, the initial draw is uniform; then from period to period, firm productivity stays constant with probability 0.9 and it changes to a neighboring state with equal probability. We use the urn–ball matching function $m(\lambda) = 1 - e^{-\lambda}$ and set $F(L) = L^{0.7}$, $f = 0.3$, $b = 1.1$, $\delta_0 = s_0 = 0.01$ and $K = 4.27$. For these parameter values the social value of an unemployed worker is $\mu = 0.2$, the unemployment rate is 2.8% and the unemployment hazard rate is at 57%. Optimal exiting prescribes that firms with the lowest productivity $x = 1.2$ leave the market whereas all other firms stay. Figure 5 shows value and policy functions (separations and recruiting strategies) for firms in the five different productivity states. Hiring and separation policies for firms with $x = 1.2$ are irrelevant as these firms leave the market.

To find the stationary firm distribution, we simulate the evolution of 1000 entrant firms over time, using the stationary policy functions describing firm growth and exiting. We set the maximum firm age at 200 periods and let all firms exit at age 201 if they have not exited earlier. This gives a total of 43,094 observations; aggregate employment and unemployment are calculated to be $E = 52,454.5$ and $U = 1,514.9$, so we normalize the labor force at $E+U$. Equivalently, for a given size of the labor force, the number of entrant firms must be rescaled accordingly. We find that total job creation (which equals total job destruction in the stationary firm distribution) is at 1,619.9 (3.1% of employment). About 32.5% of all job creation happens at entering firms and about 46% of job destruction is due to firm exit.
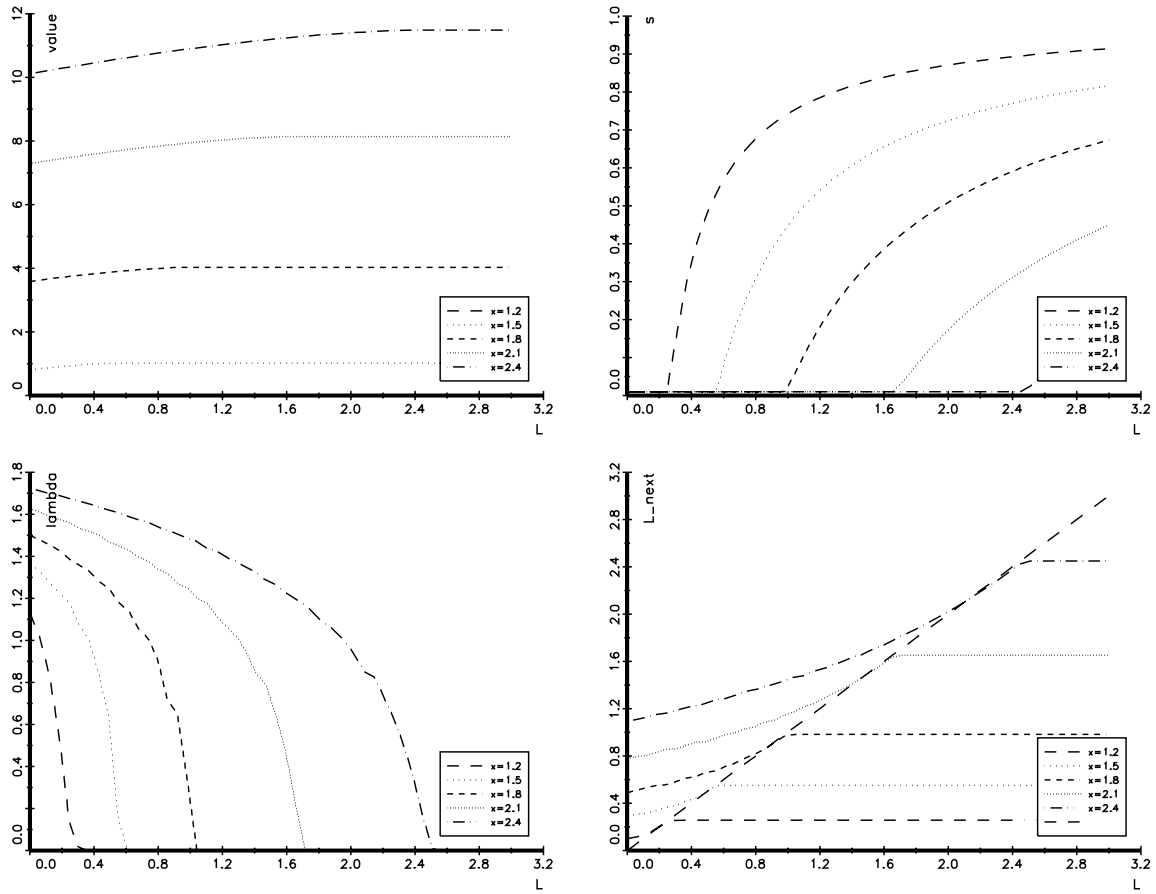
Figure 2: The firms' value functions (upper left), policy functions for s and λ (upper right and lower left), and the transition maps for employment (lower right).

Figure 5 shows how job creation and job destruction is distributed between various firm growth ranges.

To study the impact of aggregate shocks, we presuppose that entry is positive in all periods and proceed as suggested in Section 4.2. Let the aggregate productivity state be drawn from $Z = \{0.99, 1, 1.01\}$, and consider a symmetric Markov process which stays in the same state with 70% probability. The numerical solution to (24) and (25) yield then $\mu_1 = 0.215$, $\mu_2 = 0.233$ and $\mu_3 = 0.250$, and we obtain policy functions depending on $(L, x, z) \in [0, \overline{L}] \times X \times Z$. Starting with the stationary
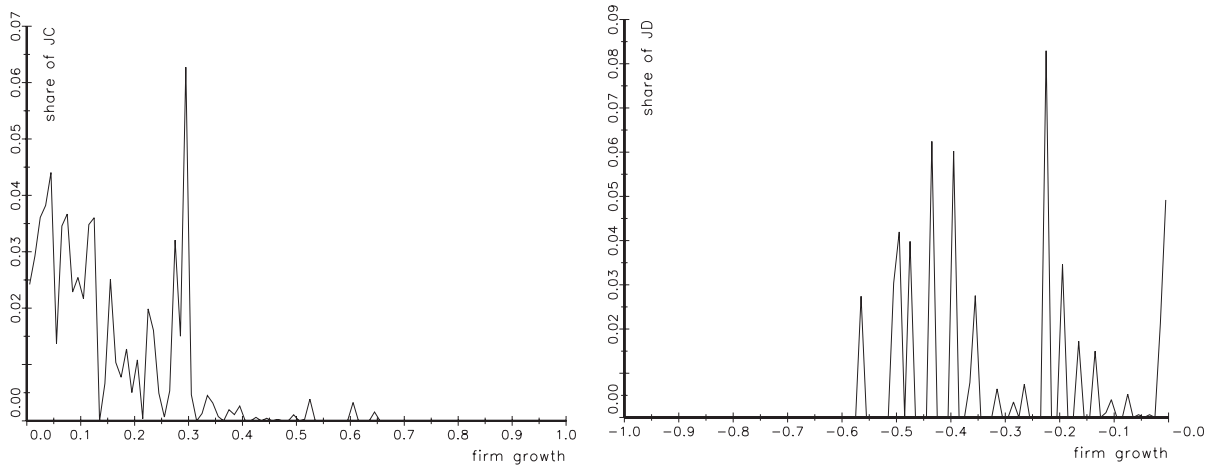
Figure 3: The distribution of job creation and job destruction. Percentages are for intervals of firm growth rates of size 0.01 (ranging from 0 to 1 for job creation and from -1 to 0 for job destruction).

firm distribution obtained before, we simulate the evolution of these firms over 50 periods using the numerical policy functions. At every point in time, the number of firm entrants is calculated as the residual of the economy's resource constraint (whenever positive). That is, in state $z \in Z$, every entrant with idiosyncratic productivity $x \in X$ attracts $\lambda(0, x, z)V(0, x, z)$ unemployed workers. The sum of these workers together with those workers who are either employed or who search for jobs at one of the existing firms must sum up to the constant size of the labor force. Figure 5 shows exemplary time series for employment, firm entry, job creation and job destruction.

6   Conclusion

The introduction of multi-worker firms into labor search models bridges the separate literatures on firm dynamics and labor search. It has the potential to address issues in both fields, and most importantly to create new insights into the interplay between firm size, entry and exit and the levels and fluctuations in employment. This
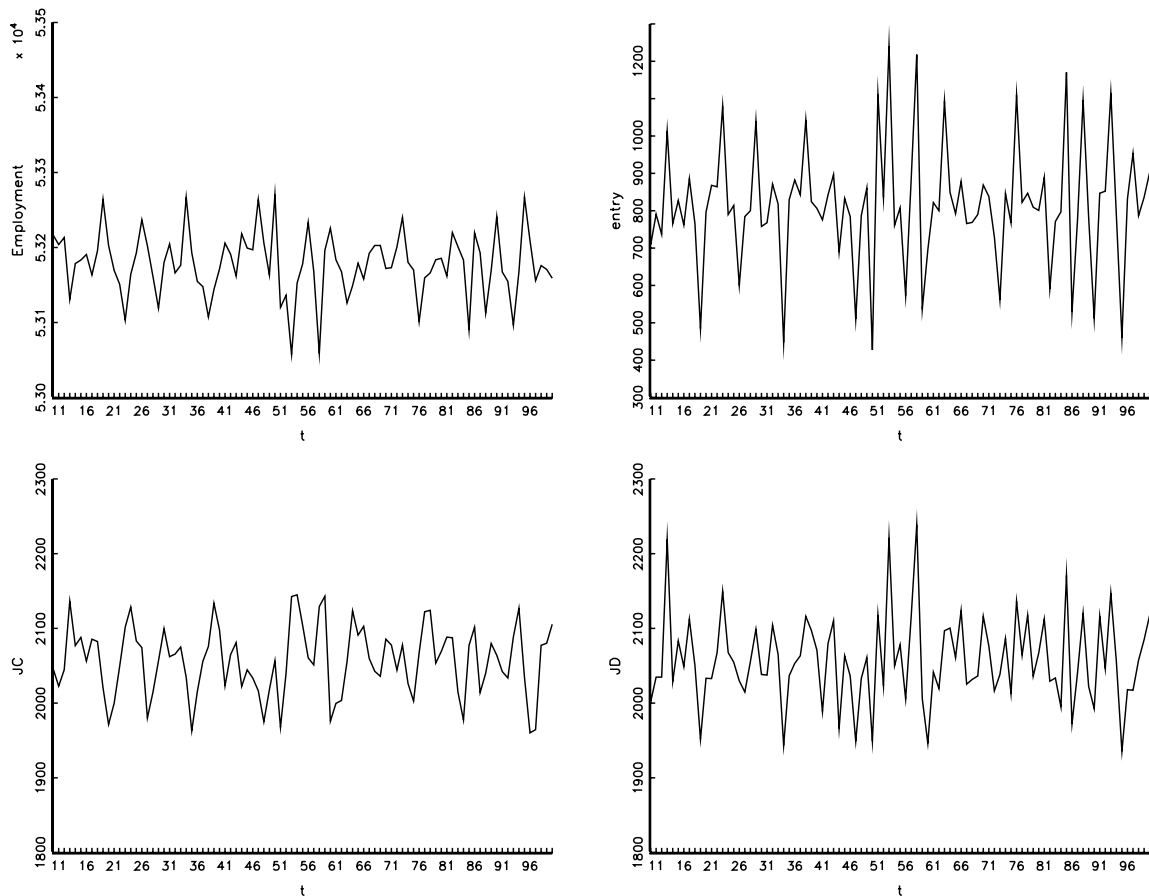
31

Figure 4: A simulated time series for employment, firm entry, job creation and job destruction when the aggregate state z fluctuates between 0.95, $z = 1$ and $z = 1.05$.

particular project proposes a wage formation process for such environments that takes into account standard competitive elements adjusted for the fact that search behavior does not allow for perfect market clearing. The model turns out to match the stylized facts regarding firm growth and pay, and implements a socially efficient allocations both in and out of steady state. It can be viewed as a benchmark against which to judge actual labor market allocations.

To conclude, it might be worthwhile to highlight a particular asymmetry between the bargaining approach and the competitive search approach to wage setting with

large firms. While the former never yields an efficient benchmark, the latter does allow to capture both efficient and inefficient scenarios. In particular, in a search environment individuals are faced with uncertainty regarding their success in finding a job. If workers are risk-averse, then treating them as if they are risk-neutral is only applicable in the benchmark case where risk is perfectly insurable through other assets. In reality some insurance through buffer savings seems possible, but full insurance might not be attainable. Lack of perfect insurance induces workers to search too much for low-paying but safe jobs, and leads to excess employment in low-productivity jobs and overall to inefficient over-employment in the absence of unemployment insurance, and they arise even in one-worker-one-firm models (see Acemoglu and Shimer (1999a)). Our environment can therefore be regarded as a benchmark not only relative to bargaining models, but also relative to alternative scenarios within the class of competitive search models.

Appendix

Proof of Proposition 1:

We have already shown that the value function can be written as $-BW + G^x(L; \rho)$ with $B = 1/[1 - \beta(1 - \eta)]$. Moreover, from (5) follows that $G^x(L; \rho)$ satisfies the recursive problem

$$
\begin{aligned}
G^x(L; \rho) \quad = \quad & \max_{(\lambda, V) \geq 0} xF(L) - C(V, L, x) - \beta[\lambda\rho + (1 - \delta)Bm(\lambda)R(\rho)]V \\
& + \beta(1 - \delta)G^x(\hat{L}; \rho) , \qquad\qquad\qquad (39) \\
& \text{s.t. } \hat{L} = L(1 - s) + m(\lambda)V , \ V \geq 0 .
\end{aligned}
$$

This problem is equivalently defined on a compact state space $L \in [0, \overline{L}]$ where $\overline{L}$ is so large that it never binds. This is possible because of the Inada condition $\lim_{L\to\infty} F'(L) = 0$. The RHS in problem (39) defines an operator T which maps a continuous function $G_0^x(L; \rho)$, defined on $[0, \overline{L}] \times [0, \overline{\rho}]$ into a continuous function $G_1^x(L; \rho) = T(G_0^x)(L; \rho)$ defined on the same domain. This operator is a contraction and it maps functions which are increasing in L and decreasing in ρ into functions with the same property. Furthermore, it maps functions which are concave in L and differentiable in (L, ρ, x) into functions with the same property. Therefore,

the unique fixed point G^x inherits all these properties. Strict concavity and strict monotonicity of F further implies that G^x is strictly increasing and strictly concave. Moreover, the fixed point must be decreasing in ρ and strictly increasing in x, which follows from differentiation of G^x with respect to ρ and x (and the assumptions on C).

Because of strict concavity of the problem, policy functions $\lambda^x(L)$ and $V^x(L, \lambda^x(L))$ exist. Properties of $V^x(L, \lambda)$ follow from the discussion in the text. To see how $\lambda^x(L)$ depends on L, differentiate the first–order condition (7) at a point where $\lambda^x(L) > 0$ and $V^x(L, \lambda^x(L)) > 0$ to obtain

$$\frac{d\lambda^x(L)}{dL} = -\frac{G^{x''}(\hat{L})\left[(1-s) + m(\lambda)\frac{dV^x}{dL}\right]}{\frac{\rho}{1-\delta}\frac{m''(\lambda)}{(m'(\lambda))^2} + G^{x''}(\hat{L})\left[m'(\lambda)V^x + m(\lambda)\frac{dV^x}{d\lambda}\right]} < 0 \ .$$

This shows that λ^x is decreasing in L. To verify the last claim, observe first that (8) and $C_{13}'' \geq 0$ imply that V^x is (weakly) decreasing in x. Furthermore, since the operator T maps a function G_0^x whose derivative $G_0^{x'}$ is increasing in x into a function G_1^x whose derivative $G_1^{x'}$ is strictly increasing in x, the unique fixed point G^x also has a marginal product which is strictly increasing in x. Hence, differentiation of (7) with respect to x implies that

$$\frac{d\lambda^x(L)}{dx} = -\frac{G^{x''}(\hat{L})m(\lambda)\frac{dV^x}{dx} + \frac{dG^{x'}}{dx}}{\frac{\rho}{1-\delta}\frac{m''(\lambda)}{(m'(\lambda))^2} + G^{x''}(\hat{L})\left[m'(\lambda)V^x + m(\lambda)\frac{dV^x}{d\lambda}\right]} > 0 \ .$$

This shows that $\lambda^x(L)$ is increasing in x. $\qquad\square$

Lemma 1: Equation (13) has a unique steady state solution $\lambda^* > 0$ if, and only if,

$$h < \frac{\beta(1-\delta)\overline{m}}{1 - \beta(1-\eta)} \ , \tag{40}$$

with $\overline{m} = \lim_{\lambda \to \infty} m(\lambda) - \lambda m'(\lambda) > 0$. Under this condition, any sequence $\lambda_t > 0$ satisfying this equation converges to λ^*.

Proof: A steady state λ^* must satisfy the condition

$$\beta\rho[m(\lambda) - \lambda m'(\lambda)] = \frac{\rho h[1 - \beta(1-\eta)]}{1 - \delta} + [Rh + c]m'(\lambda) \ . \tag{41}$$

The LHS is strictly increasing and goes from 0 to $\beta\rho\overline{m}$ as λ goes from 0 to $+\infty$. The RHS is decreasing in λ with limit $\rho h(1 - \beta(1 - \eta))/(1 - \delta)$ for $\lambda \to \infty$. Hence, a unique steady state λ^* exists iff (40) holds. Furthermore, differentiation of (13) at λ^* implies that

$$\left.\frac{d\lambda_{t+1}}{d\lambda_t}\right|_{\lambda^*} = \frac{h}{\beta(1 - \delta)m(\lambda^*) + h\beta(1 - \eta)} \ ,$$

which is positive and smaller than one iff

$$h < \frac{\beta(1 - \delta)m(\lambda^*)}{1 - \beta(1 - \eta)} \ .$$

But this inequality must be true because (41) implies

$$h = \frac{\beta\rho[m(\lambda^*) - \lambda^* m'(\lambda^*)] - cm'(\lambda^*)}{\dfrac{\rho[1 - \beta(1 - \eta)]}{1 - \delta} + Rm'(\lambda^*)} < \frac{\beta(1 - \delta)m(\lambda^*)}{1 - \beta(1 - \eta)} \ .$$

Therefore, the steady state λ^* is locally stable. Moreover, equation (13) defines a continuous, increasing relation between λ_{t+1} and λ_t which has only one intersection with the 45–degree line. Hence, $\lambda_{t+1} > \lambda_t$ for any $\lambda_t < \lambda^*$ and $\lambda_{t+1} < \lambda_t$ for any $\lambda_t > \lambda^*$, which implies that λ_t converges to λ^* from any initial value $\lambda_\tau > 0$. $\quad\square$

Proof of Proposition 2:

It remains to prove existence and uniqueness. From Proposition 1 follows, that the entrant's value function $J^x(0, 0)$ is decreasing and continuous in ρ. Hence the expected profit prior to entry,

$$\Pi^*(\rho) \equiv \sum_{x \in X} \pi(x) J^x(0, 0)$$

is a decreasing and continuous function of ρ. Moreover, the function is strictly decreasing in ρ whenever it is positive. This also follows from the proof of Proposition 1 which shows that $G^x(0; \rho)$ is *strictly* decreasing in ρ when the new firm x recruits workers ($V(0, x) > 0$). If no new firm recruits workers, expected profit of an entrant cannot be positive. Hence, equation (14) can have at most one solution for any $K > 0$. Such a solution exists provided that K is sufficiently small and $F'(0)$ is sufficiently large. To see this, when $F'(0)$ is sufficiently large, $\Pi^*(0)$ is strictly positive: some entrants will recruit workers since the marginal product $G^{x'}(m(\lambda)V; \rho)$ is sufficiently large relative to the cost of recruitment and relative to the wage cost which are, for

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$\rho = 0$, equal to $m(\lambda)Vb$ (see equation (39)). But when $\Pi^*(0) > 0$, a sufficiently small value of K guarantees that (14) has a solution since $\lim_{\rho \to \infty} \Pi^*(\rho) = 0$. □

Proof of Proposition 3:

Because the social planner's problem (16) is concave, it suffices to show that the stationary competitive search equilibrium satisfies the first–order conditions of this problem. We denote by $S_{N,a}$ the derivative of S with respect to N_a and by $S_{L,a,x}$ the derivative of S with respect to L_a^x. The multiplier on the resource constraint is $\mu \geq 0$. First–order conditions with respect to N_0, V_a^x, and λ_a^x, $a \geq 0$, are

$$\sum_{x \in X} \pi(x)\Big[xF(0) - C(V_0^x, 0, x)\Big] - K + \beta(1 - \delta)S_{N,1} - \mu \sum_{x \in X} \pi(x)\lambda_0^x V_0^x = 0 , \quad (42)$$

$$-N_a\pi(x)\Big[xF'(L_a^x) + C_a'(V_a^x, L_a^x, x) + \mu\lambda_a^x\Big] + \beta S_{L,a+1,x}m(\lambda_a^x) \leq 0 , \quad V_a^x \geq 0 , \quad (43)$$

$$\beta S_{L,a+1,x}m'(\lambda_a^x) - \mu N_a\pi(x) = 0 . \quad (44)$$

Here condition (43) holds with complementary slackness. The envelope conditions are, for $a \geq 1$ and $x \in X$,

$$S_{L,a,x} = N_a\pi(x)\Big[xF'(L_a^x) - C_2'(V_a^x, L_a^x, x) - b - \mu\Big] + \beta(1 - s)S_{L,a+1,x} , \quad (45)$$

$$S_{N,a} = \sum_{x \in X} \pi(x)\Big[xF(L_a^x) - C(V_a^x, L_a^x) - bL_a^x\Big] + \beta(1 - \delta)S_{N,a+1}$$

$$- \mu \sum_{x \in X} \pi(x)\Big(L_a^x + \lambda_a^x V_a^x\Big) . \quad (46)$$

Use (44) to substitute $S_{L,a,x}$ into (45) to obtain

$$xF'(L_{a+1}^x) - C_2'(V_a^x, L_a^x, x) - b - \mu = \frac{\mu}{\beta}\left[\frac{1}{(1 - \delta)m'(\lambda_a^x)} - \frac{\beta(1 - s)}{m'(\lambda_{a+1}^x)}\right] .$$

This equation describes the planner's optimal recruitment policy; it coincides with equation (12) for $\mu = R - b = \beta\rho$. This is intuitive: when the social value of an unemployed worker μ coincides with the surplus value that an unemployed worker obtains in search equilibrium, the firm's recruitment policy is efficient. Next substitute (44) into (43) to obtain the socially optimal vacancy creation, for $a \geq 0$ and $x \in X$,

$$C_1'(V_a^x, L_a^x, x) \geq \mu\left[\frac{m(\lambda_a^x)}{m'(\lambda_a^x)} - \lambda_a^x\right] , \quad V_a^x \geq 0 . \quad (47)$$

Again for $\mu = \beta\rho$, this condition coincides with the firm's choice of vacancy postings in competitive search equilibrium, equation (8). Lastly, it remains to verify that the social value of a jobless worker is indeed equal to $R - b$. The planner's choice of firm entry, condition (42), together with the recursive equation for the marginal firm surplus $S_{N,a}$, equation (46), shows that

$$K = \sum_{a \geq 0}[\beta(1 - \delta)]^a \sum_{x \in X} \pi(x)\Big[xF(L_a^x) - bL_a^x - C(V_a^x, L_a^x, x) - \mu(L_a^x + \lambda_a^x V_a^x)\Big] . \quad (48)$$

On the other hand, the expected profit value of a new firm is

$$\sum_{x \in X} \pi(x)J^x(0,0) = \sum_{a \geq 0}[\beta(1 - \delta)]^a \sum_{x \in X} \pi(x)\Big[xF(L_a^x) - W_a^x - C(V_a^x, L_a^x, x)\Big] .$$

Hence, the free–entry condition in search equilibrium, equation (14), coincides with condition (48) for $R = b + \mu$ if, for all $x \in X$,

$$\sum_{a \geq 0}[\beta(1 - \delta)]^a\Big[(b + \mu)L_a^x + \mu\lambda_a^x V_a^x - W_a^x\Big] = 0 . \quad (49)$$

Now after substitution of

$$L_a^x = \sum_{k=0}^{a-1}(1 - s)^{a-1-k}m(\lambda_k^x)V_k^x , \text{ and}$$

$$W_a^x = \sum_{k=0}^{a-1}(1 - s)^{a-1-k}V_k^x\Big[\frac{\rho\lambda_k^x}{(1 - \delta)B} + m(\lambda_k^x)R\Big]$$

into (49), it is straightforward to see that the equation is satisfied for $\mu = R - b = \beta\rho$.
□

Proof of Proposition 4:

The RHS in the system of equations in (22) defines an operator T that maps a sequence of bounded functions $G = (G_t)_{t \geq 0}$, with $G_t : [0, \overline{L}] \times X \times Z^t \to \mathbb{R}$ such that $\|G\| \equiv \sup_t \|G_t\| < \infty$, into another sequence of bounded functions $\tilde{G} = (\tilde{G}_t)_{t \geq 0}$ with $\|\tilde{G}\| = \sup_t \|\tilde{G}_t\| < \infty$. Here $\overline{L}$ is sufficiently large such that the bound $\hat{L} \leq \overline{L}$ does not bind for any $L \in [0, \overline{L}]$. This follows from the Inada condition for F: the marginal product of an additional worker $xzF'(\hat{L}) - b$ must be negative for any $x \in X$, $z \in Z$, for any $\hat{L} \geq \overline{L}$ with sufficiently large $\overline{L}$; hence no hiring will occur

beyond $\overline{L}$. Because the operator satisfies Blackwell's sufficient conditions, it is a contraction in the space of bounded function sequences G. Hence, the operator T has a unique fixed point which is a concave function of L. This implies that the solutions to problem (22) are characterized by its first–order conditions.

To see that the two problems (20) and (22) lead to the same choices of δ, s, λ and V for any given firm, rewrite the Lagrange function of problem (20) in the following way, with multipliers $\beta^t \psi(z^t)\mu(z^t) \geq 0$ on constraints (21),

$$
\begin{aligned}
\mathcal{L} \;=\; \max \sum_{t\geq 0, z^t} \beta^t \psi(z^t) \Bigg\{ & -KN_0(z^t) + \sum_{a\geq 0, x^a} N(x^a, z^t)\Big[x_a z_t F(L(x^a, z^t)) - bL(x^a, z^t) \\
& -f - C(V(x^a, z^t)) - \mu(z^t)\Big[(1 - s(x^a, z^t))L(x^{a-1}, z^{t-1}) + \lambda(x^a, z^t)V(x^a, z^t)\Big]\Big]\Bigg\}
\end{aligned}
$$

The planner's policy is obtained by maximization of this Lagrange function with respect to $\delta(.)$, $s(.)$, $\lambda(.)$, $V(.)$, and $N_0(.)$. Using the sequential formulation of the recursive problem (22), the maximum of the Lagrange function is the same as the sum of the social values of entrant firms and the social values of the firms that already exist at $t = 0$, namely,

$$
\begin{aligned}
\mathcal{L} \;=\; \max_{N_0(.)} & \sum_{t, z^t} \beta^t \psi(z^t) N_0(z^t)\Big[-K + \sum_{x} \pi_0(x) G_t(0, x, z^t)\Big] \\
& + \sum_{z\in Z} \psi(z^0) \sum_{a\geq 1, x^{a-1}} \pi(x_a|x_{a-1}) G_0(L(x^{a-1}, .), x_a, z^0) \;.
\end{aligned}
$$

This also proves that the complementary–slackness condition (23) describes optimal entry. $\qquad\square$

Proof of Proposition 5:

Part (a) Solving (22) in the stationary case involves to find a single value function $G(L, x, \mu)$. Application of the contraction mapping theorem implies that such a solution exits, is unique, and is continuous and non–increasing in μ and strictly decreasing in μ when $G(.) > 0$.

Therefore, the function $\Gamma(\mu) \equiv \sum_x \pi_0(x) G(0, x, \mu) \geq 0$ is continuous, strictly decreasing when positive, and zero for large enough μ. Furthermore, when f and b are

sufficiently small, $\Gamma(0) > 0$; hence when $K > 0$ is sufficiently small, there exists a unique $\mu \geq 0$ satisfying equation (23).

Part (b) For any given vector $(\mu_1, \ldots, \mu_n) \in R_+^n$, the system of recursive equations (22) has a unique solution $G = (G^i)$. Again this follows from the application of the contraction–mapping theorem. Furthermore, G is differentiable in μ, and all elements of the Jacobian $(dG^i/(d\mu_j))$ are non–positive. The RHS of (22) defines an operator mapping a function $G^i(L, x, \mu)$ with a strictly diagonally dominant Jacobian matrix $(dG^i/(d\mu_j))$ into another function $\tilde{G}_j$ whose Jacobian matrix $(d\tilde{G}_i/(d\mu_j))$ is diagonally dominant. This follows since the transition matrix $\psi(z_j|z_i)$ is strictly diagonally dominant and since all elements of $(d\tilde{G}_i/(d\mu_j))$ have the same (non–positive) sign. Therefore, the unique fixed point has a strictly diagonally dominant Jacobian. Now suppose that $(z_1, \ldots, z_n)$ is close to $(z, \ldots, z)$ and consider the solution $\mu_1 = \ldots = \mu_n = \mu$ from part (a). Since the Jacobian matrix $dG^i(0, x, \mu)/(d\mu_j)$ is strictly diagonally dominant, it is invertible. By the implicit function theorem, a unique solution to equation (23) exits. $\qquad\square$

Proof of Proposition 6: The proof proceeds in two steps. First, substitute the participation constraint (36) into the firm's problem and make use of the contracts' recursive equations (29) to show that the firms' recursive profit maximization problem is identical to the maximization of the social surplus of a firm. Second, show that the competitive equilibrium is socially optimal.

First, define the social surplus of a firm with history (x^a, z^t) and with predetermined contracts and employment levels as follows:

$$G_a\Big[(C_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] \equiv J_a\Big[(C_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t\Big] \tag{50}$$

$$+ \sum_{\tau=0}^{a-1} L_\tau\Big[W(C_\tau, x^a, z^t) - U(z^t)\Big] .$$

Without loss of generality, the participation constraint in (36) always binds, and the wage in the hiring period can be expressed as

$$w_a(x^a, z^t) = b + \beta E_{z_t} U(z^{t+1}) + \frac{\lambda\rho(z^t)}{m(\lambda)} - \beta \sum_{x_{a+1}} \sum_{z_{t+1}} \pi(x_{a+1}|x_a)\psi(z_{t+1}|z_t)W(C_a, x^{a+1}, z^{t+1}) .$$

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Now substitute this equation, (29) and (33) into (50), and write

$$S \equiv \left[(C_\tau)_{\tau=0}^{a-1}, (L_\tau)_{\tau=0}^{a-1}, x^a, z^t \right] \text{ and } \hat{S} \equiv \left[(C_\tau)_{\tau=0}^{a}, (\hat{L}_\tau)_{\tau=0}^{a}, x^{a+1}, z^{t+1} \right],$$

with $\hat{L}_\tau$ as defined in (33), to obtain

$$
\begin{aligned}
G_a(S) =\ & \max_{\delta,\lambda,V,C_a} (1-\delta) \Bigg\{ x_a z_t F(\hat{L}) - f - C(V) - \sum_{\tau=0}^{a-1} \frac{1 - \varphi_\tau(x^a, z^t)}{1-\delta} L_\tau w_\tau(x^a, z^t) \qquad (51) \\
& - m(\lambda)V \left[b + \beta E_{z_t} U(z^{t+1}) + \frac{\lambda \rho(z^t)}{m(\lambda)} \right] \\
& + \beta E_{x_a} E_{z_t} \left[J_{a+1}(\hat{S}) + m(\lambda)VW(C_a, x^{a+1}, z^{t+1}) \right] \Bigg\} \\
& + \sum_{\tau=0}^{a-1} L_\tau (1 - \varphi_\tau(x^a, z^t)) \left[w_\tau(x^a, z^t) - U(z^t) + \beta E_{x_a} E_{z_t} W(C_\tau, x^{a+1}, z^{t+1}) \right] \\
=\ & \max_{\delta,\lambda,V,C_a} (1-\delta) \Bigg\{ x_a z_t F(\hat{L}) - f - C(V) - \rho(z^t)\lambda V - b\hat{L} - \rho(z^t) \sum_{\tau=0}^{a-1} L_\tau(1 - \varphi_\tau(x^a, z^t)) \\
& + \beta E_{x_a} E_{z_t} \left[J_{a+1}(\hat{S}) + \sum_{\tau=0}^{a} \hat{L}_\tau \left(W(C_\tau, x^{a+1}, z^{t+1}) - U(z^{t+1}) \right) \right] \Bigg\} \\
=\ & \max_{\delta,\lambda,V,C_a} (1-\delta) \Bigg\{ x_a z_t F(\hat{L}) - f - C(V) - \rho(z^t) \left[\lambda V + \hat{L} - m(\lambda)V \right] - b\hat{L} \\
& + \beta E_{x_a} E_{z_t} G_{a+1}(\hat{S}) \Bigg\} .
\end{aligned}
$$

Here maximization is subject to (33) and (34), and the second equation makes use of (31). This shows that the firm solves a surplus maximization problem which is identical to the one of the planner specified in (22) provided that $\rho(z^t) = \mu(z^t)$ holds for all z^t, where μ is the social value of an unemployed worker as defined in section 4.2. The only difference between the two problems is that the firm commits to state–contingent and cohort–specific separation probabilities, whereas the planner chooses in every period an identical separation probability for all workers (and he clearly has no reason to do otherwise). Nonetheless, both problems must have the same solution: they are dynamic optimization problems of a single decision maker in which payoff functions are the same (with $\rho(z^t) = \mu(z^t)$) and the decision sets are the same. Further, time inconsistency is not an issue since there is no strategic interaction

and since discounting is exponential. Hence solutions to the two problems, with respect to firm exit, layoffs and hiring strategies, are identical. In both problems the decision maker could principally treat different cohorts differently. Because our notation for the planner solution did not capture that possibility, there is also no reason for competitive search to produce such an outcome. Nonetheless, there can be equilibria where different cohorts have different separation probabilities, but these equilibria must also be socially optimal because they maximize social firm value.

It remains to verify that worker surplus $\rho(z^t)$ in the competitive search equilibrium coincides with the social value of a worker in the planning solution. When $\mu(z^t) = \rho(z^t)$, $G_0(x, z^t)$ as defined in (50) coincides with $G_0(0, x, z^t)$, as defined in (22). Hence, the free–entry condition (37) coincides with the condition for socially optimal firm creation (23). Because of labor market clearing (38), the planner's resource constraint (21) is also satisfied. Since the allocation in competitive search equilibrium satisfies the first–order conditions and the constraints of the planner's problem, it is socially optimal. $\qquad \square$

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