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Price Level Uniformity in a Random Matching Model with Perfectly Patient Traders

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Abstract

This paper shows that one of the defining features of Walrasian equilibrium—law of one price—characterizes equilibrium in a non-Walrasian environment of (1) random trade matching without double coincidence of wants, and (2) strategic, price-setting conduct. Money is modeled as perfectly divisible and there is no constraint on agents' money inventories. In such an environment with discounting, the endogenous heterogeneity of money balances among agents implies differences in marginal valuation of money between distinct pairs of traders, which raises the question whether decentralized trade would typically involve price dispersion. We investigate the limiting case in which agents are patient, in the sense that they have overtaking-criterion preferences over random expected-utility streams. We show that in this case the "law of one price" holds exactly. That is, in a stationary Markov monetary equilibrium, all transactions endogenously must occur at a single price despite the decentralized organization of exchange. The result is in the same spirit as the work of Gale (1986a, b) on bargaining and competition, although the model differs from Gale's in some significant respects.

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1. Introduction

In a decentralized-trading environment, difference among traders creates a real possibility that decentralized trade could typically involve price dispersion. For example, it is plausible that traders with low money balances would be more desperate to acquire money than those with high balances, and would therefore offer a lower price to increase the likelihood of the offer being accepted. Indeed, Camera and Corbae (1999), Aiyagari, Wallace and Wright (1996), and Kamiya and Sato (2001) calculate examples of price-dispersion equilibrium in such environments.

On the other hand, there is a presumption that the "law of one price" should hold in a market environment where trade is decentralized but nevertheless competitive. One view of the counterexamples just cited is that, in a random-matching environment where traders maximize discounted expected utility, trade cannot be competitive. That is, if traders i and j are currently matched and i desires to consume the good that j can produce, then j is the only trader in the entire economy who can enable i to consume that good without discounting the utility derived from it, because time must elapse before i can meet another potential seller. That is, j has some monopoly power with respect to his current trading partner, i. Equilibrium price dispersion can be viewed as a consequence of such monopoly power.

One might conjecture, then, that price dispersion would be negligible in a pairwise-matching economy where sellers have little monopoly power, either because buyers' discount factor is very close to unity or because successive matches are very closely spaced in time. Camera and Corbae (1999) have proved a limit theorem to the effect that a special class of monetary equilibrium in a somewhat different random-matching environment must converge to single-price equilibrium as the discount factor approaches unity. This paper concerns another conjecture in the same spirit: that there should be no price dispersion—that is, that the "law of one price" should hold exactly—in a random-matching environment where traders have overtaking-criterion preferences. Restricting attention to stationary Markov equilibria, we prove this fact in this paper.

Our result is in the same spirit as the work of Gale (1986a, b) on bargaining and competition. There is a number of technical differences between Gale's model and ours, of which two are noteworthy. Gale models sequential exchange of claims on consumption goods, but consumption does not actually occur until after a trader has left the market; we model a trader as consuming immediately when a trade has been transacted, and then remaining in the market to transact further trades. Gale assumes that each trader has positive marginal utility for all goods, which implies that a double coincidence of wants

exists in each trade meeting and money would be inessential; our specification rules out double coincidence of wants entirely to focus on the role of money.[1]

2. The Environment

Economic activity occurs at dates $0, 1, 2, \ldots$. Agents are infinitely lived, and they are nonatomic. For convenience, we assume that the measure of the set of all agents is one. Each agent has a type in $(0, 1]$. The mapping from the agents to their types is a uniformly distributed random variable, independent of all other random variables in the model. Similarly, there is a continuum of differentiated goods, each indexed by a number $j \in (0, 1]$. These goods are perfectly divisible but nonstorable. Each agent of type i receives an endowment of one unit of "brand" i good in each period. Each agent consumes his own endowment and half of the brands in the economy; agent i consumes goods $j \in [i,\, i + \frac{1}{2}] (\text{mod } 1)$ (for example, agent 0.3 consumes goods $j \in [0.3, 0.8]$, and agent 0.7 consumes goods $j \in [0.7, 1] \cup (0, 0.2]$). He prefers other goods in his consumption range to his own endowment good; while consumption of his endowment yields utility c per unit, consumption of any other good in his feasible range yields utility u per unit, and $u > c > 0$. In addition to the consumption goods, there is a perfectly divisible and durable fiat-money object. An agent can costlessly hold any quantity of money. The total nominal stock of money remains constant at M units per capita. We assume that agents do not discount future utility. Their preferences are characterized by an overtaking criterion with respect to expected utility, which will be formalized below.

Agents randomly meet pairwise each period. By the assumed pattern of specialization in production and consumption, there is no double coincidence of wants in any pairwise meeting.[2] Each agent meets an owner of one of his consumption goods with probability one-half, and a consumer of his endowment good with probability one-half. So, every meeting is between a potential buyer and seller.

Consumption goods cannot be used as a commodity money because they are non-storable, so money is the only medium of exchange available. An agent is characterized by his type and the amount of money he holds. Within a pairwise meeting, each agent observes the other's type, but not the trading partner's money holdings and trading history. They cannot communicate about this information either. However, the economy-wide money-holdings distribution is common knowledge. For simplicity, we assume that each transaction

[1] Gale points out that his parametrization can be relaxed, but not sufficiently broadly to cover our case.

[2] Strictly speaking, there is a double coincidence of wants only when types i and j are matched, with $i \equiv j + 1/2 \pmod 1$. Such a match occurs with probability zero. Hence, we ignore this possibility.

occurs according to the following simultaneous-move game. The potential buyer submits a bid specifying a maximum price and also a quantity that he is willing to buy at any price weakly below that maximum price. And the potential seller submit an offer specifying the price at which she is willing to sell and the quantity she will sell at that price. Trade occurs if and only if the bid price is at least as high as the offer price. In that case, the buyer pays with money at the seller's offer price, and gets the smaller of the bid and offer quantities of the seller's endowment good.

3. The Definition of Equilibrium

The domain of agents' money holdings is $\mathbb{R}_+$. Let Δ be the space of countably additive probability measures on $\mathbb{R}_+$. Let $\mu_t \in \Delta$ denote the money-holdings distribution of the environment at date t.

At each date, the set of agents is randomly partitioned into pairs. Within each pair, one of the agents desires to consume the other's endowment. Thus, a bid and offer are associated with each pair.

Now we provide an intuitive discussion of the distributions of bids and offers, and we state some formal assumptions about those distributions. Our assumptions are in the spirit if a "continuum law of large numbers."[3] For each random partition π of the agents into buyer-seller pairs at date t, there is a sample distribution B_t^π of bids and a sample distribution O_t^π of offers. We assume that these sample distributions do not depend on the partition. That is, these are bid and offer distributions B_t and O_t such that for all partition π, $B_t^\pi = B_t$ and $O_t^\pi = O_t$. Moreover, because each agent has a trading partner assigned at random, the probability distribution of the trading partner's bid and offer should be identical to the sample distribution. That is, B_t and O_t are the probability distributions of bid and offer respectively that are received at date t by each individual agent, as well as being the sample distribution in each random pairing of the population of agents.

Now let the probability space $(\Omega, \mathcal{B}, P)$ represent the stochastic process of encounters faced by a generic agent. This agent faces a sequence ω of random encounters, one at each date. His date-t encounter, with some agent of type j, is characterized by her trading type (buyer or seller) in the meeting and her bid/offer price and quantity. Denote the trading partner's characteristics by $\omega_t = (\omega_{t1}, \omega_{t2}, \omega_{t3})$,

If the trading partner is a buyer, $\omega_{t1} = b$, $\omega_{t2} =$ bid price, $\omega_{t3} =$ bid quantity
If the trading partner is a seller, $\omega_{t1} = s$, $\omega_{t2} =$ offer price, $\omega_{t3} =$ offer quantity.

[3]That is, we believe that they are logically consistent with the results from probability theory that we will apply in our analysis, although they cannot be derived from those results. See Green (1994) and Gilboa and Matsui (1992) for further discussion.

The encounters $\{\omega_t\}_{t=0}^{\infty} \equiv \omega$ are independent across time. Ω is the set of all possible sequences of encounters that an arbitrary agent in the economy faces.

At each date t, pairwise meetings are independent across the population. That is, for each agent, ω_{t1} follows a *Bernoulli* distribution, a potential buyer's bid $(\omega_{t2},\, \omega_{t3})$ is drawn from the bid distribution B_t, and a potential seller's offer $(\omega_{t2},\, \omega_{t3})$ is drawn from the offer distribution O_t. For $t \geq 1$, let $\mathcal{B}_t$ be the smallest σ-algebra on Ω that makes the first $t-1$ coordinates, $\omega^{t-1} = (\omega_0, \omega_1, \ldots, \omega_{t-1})$, measurable, and $\mathcal{B}_0 = \{\phi, \Omega\}$. Let P_t be the probability measure defined on $\mathcal{B}_t$. Then, for all $t \geq 0$, $x \in \mathbb{R}_+$, and $y \in [0,1]$,

$$P_t\{\omega_{t1} = b\} = P_t\{\omega_{t1} = s\} = \frac{1}{2} \tag{1}$$

$$P_t\Big\{\omega_{t2} \leq x,\, \omega_{t3} \leq y \,|\, \omega_{t1} = b\Big\} = B_t(x,y) \tag{2}$$

$$P_t\Big\{\omega_{t2} \leq x,\, \omega_{t3} \leq y \,|\, \omega_{t1} = s\Big\} = O_t(x,y). \tag{3}$$

Define $\mathcal{B} = \mathcal{B}_\infty$ and $P = P_\infty$.

We are going to focus on symmetric equilibrium at which agents are anonymous, and an agent's strategy is only a function of his own trading history and initial money holdings. In particular, the trading strategy does not depend on an agent's type. Let σ be the trading strategy of a generic agent of type i with initial money holdings η_0. His date-t strategy $\sigma_t \equiv (\sigma_{t1},\, \sigma_{t2}, \sigma_{t3},\, \sigma_{t4})$ specifies his bid and offer as a function of his initial money holdings and his encounter history ω. The strategy σ_t is measurable with respect to $\mathcal{B}_t$. The bid $(\sigma_{t1},\, \sigma_{t2})$ is the maximum price σ_{t1} at which the agent is willing to buy and the quantity σ_{t2} that he is willing to purchase (at price no higher than σ_{t1}) if he is paired with a seller of his consumption goods. The offer $(\sigma_{t3},\, \sigma_{t4})$ represents the price σ_{t3} at which he is willing to sell and the maximum quantity σ_{t4} that he is willing to sell at price σ_{t3} if he meets a consumer of his endowment good. Because of the restriction on endowment, $\sigma_{t4} \leq 1$. As a buyer, the agent has to be able to pay his bid. Let η_t^σ denote the agent's money holdings at the beginning of date t by adopting strategy σ. (Note that η_t^σ is a function of η_0 and ω, and that it is $\mathcal{B}_t$-measurable in ω.) Then

$$\sigma_{t1}(\eta_0,\omega)\, \sigma_{t2}(\eta_0,\omega) \leq \eta_t^\sigma(\eta_0,\omega). \tag{4}$$

Given the agent's initial money holdings η_0, encounter history ω, and strategy $\sigma = \{\sigma_t\}_{t=0}^{\infty}$, his money holdings evolves recursively as follows: $\eta_0^\sigma(\eta_0,\omega) = \eta_0$ and, for $t \geq 0$,

$$\eta_{t+1}^\sigma(\eta_0,\omega) = \begin{cases} \eta_t^\sigma(\eta_0,\omega) + \sigma_{t3}(\eta_0,\omega) \min\{\sigma_{t4}(\eta_0,\omega),\, \omega_{t3}\} & \text{if } \omega_{t1} = b,\ \sigma_{t3}(\eta_0,\omega) \leq \omega_{t2} \\ \eta_t^\sigma(\eta_0,\omega) - \omega_{t2} \min\{\sigma_{t2}(\eta_0,\omega),\, \omega_{t3}\} & \text{if } \omega_{t1} = s,\ \sigma_{t1}(\eta_0,\omega) \geq \omega_{t2} \\ \eta_t^\sigma(\eta_0,\omega) & \text{otherwise} \end{cases} \tag{5}$$

Let $v_t^\sigma(\eta_0, \omega)$ denote the agent's utility achieved at date t adopting strategy σ relative to no trade that date. Then

$$v_t^\sigma(\eta_0,\omega) = \begin{cases} -c\ \min\{\sigma_{t4}(\eta_0,\omega),\ \omega_{t3}\} & \text{if } \omega_{t1} = b,\ \sigma_{t3}(\eta_0,\omega) \le \omega_{t2} \\ u\ \min\{\sigma_{t2}(\eta_0,\omega),\ \omega_{t3}\} & \text{if } \omega_{t1} = s,\ \sigma_{t1}(\eta_0,\omega) \ge \omega_{t2} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Then, strategy σ *overtakes* another strategy $\hat{\sigma}$ if for all $\eta_0 \in \mathbb{R}_+$,

$$\liminf_{t\to\infty}\ \mathsf{E}\Big[\sum_{\tau=0}^{t} v_\tau^\sigma(\eta_0,\omega) - \sum_{\tau=0}^{t} v_\tau^{\hat{\sigma}}(\eta_0,\omega)\Big] > 0 \tag{7}$$

where E is the expectation operator with respect to the probability measure P.

At the beginning of date t, given all agents' trading strategy σ_t and the initial money-holdings distribution μ_0, rational expectation requires that agents' belief regarding the bid distribution B_t and the offer distribution O_t that prevail during date-t trading confirm with the actual distributions implied by the strategy. That is, for all $x,\ y \in \mathbb{R}_+$,

$$B_t(x,y) = \int_0^\infty P_t\Big\{\omega \mid \sigma_{t1}(z,\omega) \le x,\ \sigma_{t2}(z,\omega) \le y\Big\} d\,\mu_0(z) \tag{8}$$

$$O_t(x,y) = \int_0^\infty P_t\Big\{\omega \mid \sigma_{t3}(z,\omega) \le x,\ \sigma_{t4}(z,\omega) \le y\Big\} d\,\mu_0(z). \tag{9}$$

The offer and bid distribution each extend uniquely to a corresponding measure on $\mathbb{R}_+^2$, which will also be denoted by O and B respectively. (No confusion will result, since the argument of the c.d.f. is an ordered pair of numbers while the argument of the measure is a subset of the nonnegative orthant of the plane.) That is, for all $p \ge 0$ and $q \ge 0$, $O([0,p]\times[0,q]) = O(p,q)$ and $B([0,p]\times[0,q]) = B(p,q)$.

Similarly, the money holdings distribution at the beginning of the of date t is defined as follows, for any set $A \in \mathcal{B}_t$,

$$\mu_t(A) = \int_0^\infty P_t\Big\{\omega \mid \eta_t(z,\omega) \in A\Big\} d\,\mu_0(z) \tag{10}$$

The equilibrium concept we adopt is Bayesian Nash equilibrium with overtaking criterion.

Definition 1. A Bayesian Nash equilibrium is a four-tuple $\langle \sigma, \mu_0, \{B_t\}_{t=0}^\infty, \{O_t\}_{t=0}^\infty\rangle$ that satisfies

(i) μ_0 is the initial money-holdings distribution in the environment.

(ii) Given the bid distributions $\{B_t\}_{t=0}^{\infty}$ and the offer distributions $\{O_t\}_{t=0}^{\infty}$, and given that all other agents adopt strategy σ, it is optimal for an arbitrary agent to adopt strategy σ, that is, there is no strategy that overtakes strategy σ.

(iii) For each $t \geq 0$, B_t and O_t satisfy equations (8) and (9). That is, these distributions reflect the adoption of strategy σ by all agents.

In this paper, we are interested in stationary equilibria where all the distributions (money-holdings μ_t, bid B_t, and offer O_t) are time-invariant, and agents' trading strategy is stationary Markov. A strategy σ is stationary Markov if there is a function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+^4$ such that for all $t \geq 0$, $\eta_0 \in \mathbb{R}_+$, and $\omega \in \Omega$,

$$\sigma_t(\eta_0, \omega) = \psi(\eta_t(\eta_0, \omega)) \tag{11}$$

that is, the strategy is only a function of an agent's current money holdings.

Definition 2. A stationary Markov monetary equilibrium (SMME) is a Bayesian Nash equilibrium $\langle \sigma, \mu, \{B_t\}_{t=0}^{\infty}, \{O_t\}_{t=0}^{\infty} \rangle$ such that

(i) There exist measures O, B and μ such that, for all $t \geq 0$, $B_t = B$, $O_t = O$ and $\mu_t = \mu$.

(ii) A generic agent's random process of bid prices converges almost surely to the market-wide bid distribution in the sense that, for every open subset A of $\mathbb{R}_+^2$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t<n} \chi_A(\sigma_{t1}(\eta_0, \omega), \sigma_{t2}(\eta_0, \omega)) \to B(A) \text{ , a.s. .}$$

(iii) A generic agent's sample path of offers converges almost surely to the market-wide offer distribution in the sense that (with χ_A denoting the characteristic function of a set A), for every open subset A of $\mathbb{R}_+^2$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t<n} \chi_A(\sigma_{t3}(\eta_0, \omega), \sigma_{t4}(\eta_0, \omega)) \to O(A) \text{ , a.s. .}$$

(iv) σ is a stationary Markov strategy.

(v) monetary trade is strictly better than autarky in the sense that

$$\liminf_{t\to\infty} \mathsf{E} \sum_{\tau=0}^{t} v_\tau^\sigma(\eta_0, \omega) > 0 \text{ .}$$

Items (i)–(iv) of the definition specify various aspects of stationarity, with (ii) and (iii) suggesting that agents' strategic behavior is ergodic as would be expected in a stationary equilibrium. Item (v) rules out autarkic equilibrium. Note that, since an agent can choose autarky unilaterally, no equilibrium can be worse than autarky.

Note that this equilibrium concept is a specialization of Bayesian Nash equilibrium in the space of *all* strategies defined above. That is, a SMME must be impervious to deviation to an arbitrary strategy, whether or not the deviation strategy is stationary Markov.

One particular type of SMME is the single-price equilibrium, which is defined as an equilibrium where all trades occur at the same price, say p, almost surely. That is, all traders bid to buy one unit or as much as they can afford of their desired consumption goods at price p, and offer to sell one unit of their endowment goods at price p. From the result of Green and Zhou (2000), we can conclude that a stationary single-price-p equilibria exists if the initial money holdings η_0 is distributed geometrically on the lattice defined by p, that is, on $\{0, p, 2p, \ldots\}$. Another type of potential equilibrium is the price-dispersion equilibrium, at which trades occur at different prices across different pairs of traders who have different money holdings and trading histories. In the next section, we are going to show that at a SMME, price dispersion can not occur. In other words, single-price equilibrium is the only kind of stationary Markov equilibrium that exists. Note that an autarkic equilibrium is always an single-price equilibrium.

4. Nonexistence of SMME with Price-Dispersion

Suppose that $\langle \sigma, \mu, B, O \rangle$ is a SMME defined as above. We want to show that at this equilibrium, all trades occur at the same price.

Recall that, for a probability measure $\mathcal{P}$ on a separable topological space T, the support $\text{supp}(\mathcal{P})$ of the measure is the intersection of all closed subsets $H \subset T$ such that $\mathcal{P}(H) = 1$, and that $\mathcal{P}(\text{supp}(\mathcal{P})) = 1$.

Let o^- be the infimum of the support of the marginal distribution of offer prices at which positive quantities are offered. That is,

$$o^- = \sup\{x \,|\, O([0, x) \times (0, \infty)) = 0\}. \tag{12}$$

(Note that offers to sell a zero quantity are excluded.)

We show that at equilibrium, all trades occur at price o^- almost surely. We prove this via several lemmas. Note that by the definition of the equilibrium, the bid and offer distributions B and O are generated by the equilibrium strategy σ and the money-holdings distribution μ through equations (8) and (9).

LEMMA 1. *Given that $\langle \sigma, \mu, B, O \rangle$ is a SMME, the infimum o^- of the support of the offer price distribution is positive.*

Proof. First observe that $O(\{0\} \times (0, \infty)) = 0$, since otherwise an agent would be giving away for free some quantity of endowment that he would have obtained utility from having consumed instead.

Using this fact, we show that for some $p > 0$ and $q > 0$, $(p, q) \in \text{supp}(B)$. Suppose that, to the contrary, every bid in the $\text{supp}(B)$ is either $(p, 0)$ or $(0, q)$. A bid of $(p, 0)$ results in no trade since the size of the transaction is no greater than the bid quantity, and a bid of $(0, q)$ results in no trade since $O(\{0\} \times (0, \infty)) = 0$. Since SMME is not autarkic by definition 2(v), bids of these two forms cannot constitute all of $\text{supp}(B)$. That is, for some $p > 0$ and $q > 0$, $(p, q) \in \text{supp}(B)$.

Let $p > 0$ and $q > 0$, suppose that $(p, q) \in \text{supp}(B)$, and consider offer $(p/2, q/2)$. This offer results in a sale of $q/2$ units for revenue $pq/4$ with some positive probability π, since $(p, q) \in \text{supp}(B)$. Therefore the expected revenue from making this offer is at least $\pi pq/4$. The ratio of expected revenue to expected amount of endowment sold is p.

If every offer in an open set U of the topological space $\mathbb{R}^2_+$ (with its Euclidean topology relative to $\mathbb{R}^2$) provides expected revenue less than what $(p/2, q/2)$ provides and also yields a lower ratio of expected revenue to expected amount of endowment sold than $(p/2, q/2)$ yields, then an argument involving the Strong Law of Large Numbers and Fatou's Lemma shows that $U \cap \text{supp}(O) = \emptyset$.[4] In particular, $U = [0, \pi pq/4) \times (0, \infty)$ satisfies this condition, so $U \cap \text{supp}(O) = \emptyset$. That is, $o^- \geq \pi pq/4 > 0$. ∎

The next lemma is a technical result that will be used in the proof of the subsequent lemma.

LEMMA 2. *For a given subset Q of Ω, for any $\lambda > 0$, there exist $T_\lambda \geq 0$ and $Q_\lambda \in \mathcal{B}_{T_\lambda}$ such that*

$$P(Q_\lambda \cap Q) \geq (1 - \lambda)P(Q) \quad \textit{and} \quad P(Q_\lambda \setminus Q) < \lambda P(Q). \tag{13}$$

Lemma 2 says that for a given set Q, there exist a date T_λ and a set Q_λ, measurable with respect to $\mathcal{B}_{T_\lambda}$, that is an arbitrarily close approximation of Q. The proof of Lemma 2 is in the appendix.

[4] Fatou's lemma (c.f. Royden, 1988) justifies converting an expectation of limits to a limit of expectations, as the overtaking criterion requires. See the complete proof of Lemma 3 in the appendix for a closely related argument.

Consider an agent with an initial money holdings η_0 who adopts the equilibrium strategy σ. Given that the environment is stationary, the agent's money holdings η_t^σ is markovian with full support of economy-wide distribution μ. The next lemma states that the agent's money holdings drops below any arbitrary level infinitely often.

LEMMA 3. *If $\langle \sigma, \mu, B, O \rangle$ is a SMME, then for any $\rho > 0$, an agent's money holdings $\eta_t^\sigma < \rho$ infinitely often almost surely.*

Proof. A detailed proof is given in appendix. The intuitive argument is as follows.

Suppose that, with positive probability, an agent's money holdings falls below $\rho > 0$ only finitely often. Then, by countable additivity, there must be a date T such that $P(\{\omega | \forall\, t \geq T \;\; \eta_t^\sigma \geq \rho\}) > 0$. Define $Q = \{\omega | \forall\, t \geq T \;\; \eta_t^\sigma \geq \rho\}$. By Lemma 2, for every $\lambda > 0$ there is an event Q_λ such that $P(Q_\lambda \cap Q) \geq (1-\lambda)P(Q)$ and $P(Q_\lambda \setminus Q) < \lambda P(Q)$ with Q_λ being $\mathcal{B}_{T_\lambda}$-measurable, where without loss of generality $T \leq T_\lambda$.

This situation is proved to lead to a contradiction by specifying an overtaking-criterion-preferred deviation $\hat{\sigma}$ from σ with the following intuitive form. The agent follows σ until T_λ, and after T_λ if $\omega \notin Q_\lambda$.

Choose a number $o^+ > o^-$. What $\hat{\sigma}$ prescribes for $\omega \in Q_\lambda$ is that beginning at T_λ, the agent should abstain from selling (that is, set $\sigma_{t4} = 0$) whenever σ would have set the offer price no higher than o^+. This abstention should continue until a date t when either a cumulative value of ρ in sales has been forgone, or else η_t^σ would have been less than ρ.[5] If $\omega \in Q$, then eventually the cumulative value of forgone sales will equal ρ almost surely (conditional on $Q \cap Q_\lambda$). Because the price at which these sales are forgone does not exceed o^+, forgoing the sales provides at least ρ/o^+ of the agent's endowment for consumption. Asymptotically, then, this consumption adds $c(\rho/o^+)P(Q \cap Q_\lambda)$ to the sum of expected utilities by which overtaking is calculated. Note that, by choice of Q_λ, $c(\rho/o^+)P(Q \cap Q_\lambda) \geq c(\rho/o^+)(1-\lambda)P(Q)$.

If $\omega \notin Q$, then the agent will learn this at the first time t when $\eta_t^\sigma < \rho$. (Note that the agent has the information required to know what η_t^σ would have been, although he does not play strategy σ.) When this happens (that is, in the event $Q_\lambda \setminus Q$), $\hat{\sigma}$ prescribes that the agent should abstain from purchasing until ρ has cumulatively been saved. (Limited purchasing, rather than complete abstention, is required in some circumstances to prevent cumulatively saving more than ρ.) Because the forgone purchases would have been made at prices of at least o^-, no more than ρ/o^- consumption will be forgone. That amount

[5] A positive, but reduced, value of σ_{t4} may have to be set in some circumstances, in order to reduce sales below what σ would have entailed but ensure that the cumulative value of forgone sales does not exceed ρ.

of consumption, in the event $Q_\lambda \setminus Q$, asymptotically subtracts at most $u(\rho/o^-)P(Q_\lambda \setminus Q)$ from the sum of expected utilities by which overtaking is calculated. Note that, by choice of Q_λ, $u(\rho/o^-)P(Q_\lambda \setminus Q) \leq u(\rho/o^-)\lambda P(Q)$.

Except for these changes just specified, $\hat{\sigma}$ prescribes the same bids and offers as does σ.

Asymptotically, the net benefit of deviating to $\hat{\sigma}$ is at least $[c(\rho/o^+)(1-\lambda)-u(\rho/o^-)\lambda]P(Q)$. A positive value of λ can be chosen that is small enough so that this asymptotic net benefit is positive if $P(Q) > 0$. This argument by contradiction shows that $P(Q) = 0$, that is, that $\eta_t^\sigma < \rho$ infinitely often, almost surely. ∎

Now we are ready to prove the main result of the paper.

PROPOSITION. If $\langle \sigma, \mu, B, O \rangle$ is a SMME, then it is a single-price equilibrium. In other words, given o^- is the infimum of the support of the offer price distribution O_π, all trades occur at o^- almost surely.

Proof. We first show that for any arbitrary $\delta > 0$, all trades occur at price no higher than $o^- + \delta$ almost surely. This claim holds trivially if no trader offers to sell at price higher than $o^- + \delta$, i.e., $\text{supp}(O) \cap (o^- + \delta, \infty) \times (0, \infty) = \emptyset$, since trade occurs at the offer price. The more complicated case is when offers higher than $o^- + \delta$ are made with positive probability, i.e., $\text{supp}(O) \cap (o^- + \delta, \infty) \times (0, \infty) \neq \emptyset$. In this case, we show that $\text{supp}(B) \cap (o^- + \delta, \infty) \times (0, \infty) = \emptyset$. This would contradict trade occurring at prices higher than $o^- + \delta$ with positive probability.

Suppose to the contrary, $\text{supp}(B) \cap (o^- + \delta, \infty) \times (0, \infty) \neq \emptyset$. Consider a generic trader whose strategy is σ, and whose random trading partners are described by Ω. For each $\omega \in \Omega$, let $\xi^\sigma(\omega)$ denote the set of dates at which the agent bids to buy at price above $o^- + \delta$,

$$\xi^\sigma(\omega) = \{t \mid \sigma_{t1}(\eta_0, \omega) > o^- + \delta\}.$$

By assumption and Definition 2 (ii), $\xi^\sigma(\omega)$ is an infinite set almost surely. Let ε be a small positive number, which value will be chosen later. For any $n \geq 1$, let $\varphi_n^\varepsilon(\omega)$ denote the set of dates at which the agent's money holdings are below ε^n,

$$\varphi_n^\varepsilon(\omega) = \{t \mid \eta_t^\sigma(\eta_0, \omega) < \varepsilon^n\}.$$

For each $n \geq 1$, since $\varepsilon^n > 0$, by Lemma 3, $\varphi_n^\varepsilon(\omega)$ is an infinite set almost surely.

We construct a strategy $\hat{\sigma}$ as follows. For all $\omega \in \Omega$ and $t \geq 0$, the offer $(\hat{\sigma}_{t3}, \hat{\sigma}_{t4})(\eta_0, \omega)$ is identical to $(\sigma_{t3}, \sigma_{t4})(\eta_0, \omega)$. The bid $(\hat{\sigma}_{t1}, \hat{\sigma}_{t2})$, however, differs from $(\sigma_{t1}, \sigma_{t2})$ at the following dates.

1. Change the bid price from above $o^- + \delta$ to $o^- + \delta$ through the first date at which a trade that would have occurred under σ fails to occur because the changed bid is below the offer. That is, let

$$t^* = \min\{t \,|\, t \in \xi^\sigma(\omega),\, \omega_{t1} = s,\, \sigma_{t1} \geq \omega_{t2} > o^- + \delta,\, \sigma_{t2} > 0,\, \omega_{t3} > 0\}$$

and for all $t \in \xi^\sigma(\omega)$ and $t \leq t^*$, set $\hat{\sigma}_{t1}(\eta_0, \omega) = o^- + \delta$. Let X be the quantity that would have traded on date t^* under σ, $X \equiv \min\{\sigma_{t^*2},\, \omega_{t^*2}\}$, and let S be the money saved on date t^* by not buying at price above $o^- + \delta$, that is, $S = \omega_{t^*2} X > (o^- + \delta)X$. Then, at the beginning of date $t^* + 1$, $\eta^{\hat{\sigma}}_{t^*+1} = \eta^{\sigma}_{t^*+1} + S$, $\sum_{\tau=0}^{t^*}(v^{\hat{\sigma}}_\tau - v^{\sigma}_\tau) = -uX > -uS/(o^- + \delta)$.

2. Spend the money saved at date t^*, S, at a sequence of dates after t^* when the agent's money holdings are below ε^n (i.e. dates in $\varphi^\varepsilon_n(\omega)$), consecutively, for $n = 1, 2, \ldots$. Specifically, let $l_0 = t^*$, and set n=1.

   (a) For all $t > l_{n-1}$ such that $t \in \varphi^\varepsilon_n(\omega)$, set

$$\hat{\sigma}_{t1}(\eta_0, \omega) = o^- + \delta/2, \qquad \hat{\sigma}_{t2}(\eta_0, \omega) = \min\Big\{1,\, \frac{S}{o^- + \delta/2}\Big\}$$

   until a trade is accomplished with the modified strategy at date l_n. That is,

$$l_n = \min\{t \,|\, t > l_{n-1},\, t \in \varphi^\varepsilon_n(\omega),\, \omega_{t1} = s,\, \hat{\sigma}_{t1}(\eta_0, \omega) \geq \omega_{t2},\, \omega_{t3} > 0\}.$$

   Such an occasion exists because $\varphi^\varepsilon_n(\omega)$ is an infinite set, o^- is the infimum of the support of the offer price distribution, hence there exist $p_s \in [o^-, o^- + \delta/2]$ and $q_s > 0$ such that $(p_s, q_s) \in \mathrm{supp}(O)$.

   (b) Let $S = S - \omega_{l_n 2} \min\{\hat{\sigma}_{l_n 2}(\eta_0, \omega),\, \omega_{l_n 3}\}$. If $S > 0$, set $n = n + 1$ and return to step (a). Otherwise, all money saved at date t^* is spent, and the agent resume strategy $\sigma(\eta_0, \omega)$.

   This process stops in finite time almost surely given that S is finite, and transaction sizes are determined by $\{\omega_{l_n 3}\}_{n \geq 1}$, which is i.i.d. and $\{\hat{\sigma}_{l_n 2}\}_{n \geq 1}$ which is constructed to be a non-binding constraint as long as S remains positive.

Now, let us examine the utility lost and gained by adopting the modified strategy. In step 2, the utility gained is from consuming goods purchased by the S units of saved money at price no higher than $o^- + \delta/2$, hence it is no less than $uS/(o^- + \delta/2)$. However, since we modify the bid price for $t \in \varphi^\varepsilon_n(\omega)$, if the original bid price is higher than $o^- + \delta/2$, some

trades that would have occurred under strategy σ may fail to occur under strategy $\hat{\sigma}$. In each of such instances (i.e., for all $n \geq 1$), the maximum amount of goods the agent would have bought under strategy σ is ε^n/o^-. Hence, the total utility loss due to this modification is less than

$$u\sum_{n=1}^{\infty}\varepsilon^n/o^- = \frac{u}{o^-}\frac{\varepsilon}{1-\varepsilon}.$$

Combining all the gains and losses in steps 1 and 2, given that the rest of strategy σ is unchanged,

$$\begin{aligned}\sum_{\tau=0}^{\infty}(v_\tau^{\hat{\sigma}} - v_\tau^{\sigma}) &= \sum_{\tau=0}^{t^*}(v_\tau^{\hat{\sigma}} - v_\tau^{\sigma}) + \sum_{\tau=t^*+1}^{\infty}(v_\tau^{\hat{\sigma}} - v_\tau^{\sigma}) \\ &> -u\frac{S}{o^-+\delta} + u\frac{S}{o^-+\delta/2} - \frac{u}{o^-}\frac{\varepsilon}{1-\varepsilon} \\ &= u\Big(\frac{S\delta}{2(o^-+\delta)(o^-+\delta/2)} - \frac{1}{o^-}\frac{\varepsilon}{1-\varepsilon}\Big). \qquad (14)\end{aligned}$$

Let

$$\varepsilon^* \equiv \frac{S\delta o^-}{S\delta o^- + 2(o^-+\delta)(o^-+\delta/2)}.$$

By (14), for any $\varepsilon < \varepsilon^*$,

$$\sum_{\tau=0}^{\infty}(v_\tau^{\hat{\sigma}} - v_\tau^{\sigma}) > u\Big(\frac{S\delta}{2(o^-+\delta)(o^-+\delta/2)} - \frac{1}{o^-}\frac{\varepsilon}{1-\varepsilon}\Big) > 0.$$

Hence,

$$\liminf_{t\to\infty} \mathsf{E}\Big[\sum_{\tau=0}^{t} v_\tau^{\hat{\sigma}} - \sum_{\tau=0}^{t} v_\tau^{\sigma}\Big] > 0.$$

By (7), strategy $\hat{\sigma}$ overtakes strategy σ, which contradicts the assumption that σ is an equilibrium strategy. Therefore, no price higher than $o^- + \delta$ is in the support of the bid price distribution.

We have shown that for any $\delta > 0$ such that any price higher than $o^- + \delta$ is in the support of the offer price distribution, it is not in the support of the bid price distribution. Thus, no trade occurs at price above $o^- + \delta$ with positive probability. Taking $\delta \to 0$, all trades occurs at price no higher than o^- almost surely. Since o^- is the infimum of the support of the offer price distribution, all trades occur at o^- almost surely. ∎

The intuition for the above result is quite simple. If with strictly positive probability that offers are made at some $o^- + \delta > o^-$, then the optimal strategy has to be such that with probability one no such offer is accepted, since otherwise, a perfectly patient agent

can improve the strategy by switching purchases at price $o^- + \delta$ to other dates at some lower price without any loss. The lowest price possible is o^-. Hence, no trades will occur at price other than o^-.

To summarize, despite the decentralized trading arrangement, a SMME is always a single-price equilibrium, where all trades take place at the same price among all pairwise meetings, almost surely.

5. Conclusion

In this paper, we have shown that any stationary Markov monetary equilibrium in a random-matching economy with perfectly patient agents must be a single-price equilibrium. Our demonstration of this fact does not depend on showing that equilibrium is walrasian. In this respect, the demonstration is very different from the type of theorem proved by Douglas Gale (1986a, b) for non-monetary economies. We are confident that our theorem in the limit regarding perfectly patient agents can be complemented with a limit theorem, which will state that equilibrium is approximately single-price, in a suitable sense, in an economy where agents' discount factor is almost equal to unity. We note that Camera and Corbae (1999) have proved such a limit theorem regarding a special class of monetary equilibrium in a somewhat different random-matching environment.

Appendix

The Proof of Lemma 2.

For all $t \geq 0$, define a measure μ_t on $\mathcal{B}_t$ such that for any $C \in \mathcal{B}_t$, $\mu_t(C) = P(Q \cap C)$. By this definition, for any $C \in \mathcal{B}_t$, $\mu_t(C) \leq P(C)$. Therefore, there exists a density f_t such that for all $C \in \mathcal{B}_t$, $\int_C f_t \, dP = \mu_t(C)$. In fact, $f_t = \mathsf{E}[\chi_Q \,|\, \mathcal{B}_t]$ where $\chi_Q(\omega) = 1$ if $\omega \in Q$ and $\chi_Q(\omega) = 0$ if $\omega \notin Q$, since for any $C \in \mathcal{B}_t$,

$$\int_C \mathsf{E}[\chi_Q \,|\, \mathcal{B}_t] \, dP = \int_{C \cap Q} \chi_Q \, dP + \int_{C \setminus Q} \chi_Q \, dP = P(C \cap Q) = \mu_t(C) = \int_C f_t \, dP.$$

Therefore, $\{f_t\}_{t=0}^{\infty}$ is a martingale. By Martingale convergence theorem, $f_t \to \chi_Q$ a.s.. Furthermore, $\mathsf{E}[f_t - \chi_Q] \to 0$.

For any $\lambda > 0$, for all $t \geq 0$, define $C_t \equiv \{\omega \,|\, f_t(\omega) \geq 1 - \lambda\}$. Then,

$$\begin{aligned} P(C_t \setminus Q) \leq \frac{1}{1-\lambda} \int_{C_t \setminus Q} f_t \, dP &= \frac{1}{1-\lambda} \Big(\int_{C_t \setminus Q} (f_t - \chi_Q) \, dP + \int_{C_t \setminus Q} \chi_Q \, dP \Big) \\ &= \frac{1}{1-\lambda} \int_{C_t \setminus Q} (f_t - \chi_Q) \, dP \end{aligned} \tag{15}$$

Furthermore, since for $\omega \notin C_t$, $f_t(\omega) < 1 - \lambda$,

$$\begin{aligned} P(Q \setminus C_t) = \int_{Q \setminus C_t} \chi_Q \, dP &= \int_{Q \setminus C_t} (\chi_Q - f_t) \, dP + \int_{Q \setminus C_t} f_t \, dP \\ &< \int_{Q \setminus C_t} (\chi_Q - f_t) \, dP + (1-\lambda) P(Q \setminus C_t) \end{aligned}$$

which implies that

$$P(Q \setminus C_t) < \frac{1}{\lambda} \int_{Q \setminus C_t} (\chi_Q - f_t) \, dP$$

therefore,

$$P(C_t \cap Q) = P(Q) - P(Q \setminus C_t) > P(Q) - \frac{1}{\lambda} \int_{Q \setminus C_t} (\chi_Q - f_t) \, dP. \tag{16}$$

Then, since $\mathsf{E}[f_t - \chi_Q] \to 0$, there is $T_\lambda \geq 0$ such that

$$\Big| \int_{C_{T_\lambda} \setminus Q} (f_t - \chi_Q) \, dP \Big| \leq \min\{\lambda(1-\lambda), \lambda^2\} P(Q). \tag{17}$$

By (15) and (16), (17) implies that

$$P(C_{T_\lambda} \cap Q) \geq (1-\lambda) P(Q) \quad \text{and} \quad P(C_{T_\lambda} \setminus Q) < \lambda P(Q). \tag{18}$$

Let $Q_\lambda \equiv C_{T_\lambda} \in \mathcal{B}_{T_\lambda}$, (18) is the result we set out to proof. ∎.

The Proof of Lemma 3.

Suppose that, with positive probability, an agent's money holdings falls below $\rho > 0$ only finitely often. Then, by countable additivity, there must be a $\varepsilon > 0$ and a date T such that $P(\{\omega | \forall\, t \geq T \;\; \eta_t^\sigma \geq \rho\}) > \varepsilon$. Define $Q = \{\omega | \forall\, t \geq T \;\; \eta_t^\sigma \geq \rho\}$. By Lemma 2, for every $\lambda > 0$ there exist $T_\lambda > T$ and an event $Q_\lambda \in \mathcal{B}_{T_\lambda}$ such that

$$P(Q_\lambda \cap Q) \geq (1-\lambda)P(Q) \quad \text{and} \quad P(Q_\lambda \setminus Q) < \lambda P(Q). \tag{19}$$

We show that a deviation strategy $\hat{\sigma}$ can overtake strategy σ, hence contradict that σ is an equilibrium strategy. Strategy $\hat{\sigma}$ is defined as follows. The agent follows σ until T_λ, and after T_λ if $\omega \notin Q_\lambda$.

Choose a number $o^+ \geq o^-$ such that there exist $p \leq o^+$, $q_b > 0$ and $q_s > 0$ such that $(o^+, q_b) \in \text{supp}(B)$ and $(p, q_s) \in \text{supp}(O)$. Such an o^+ exists (possibly equals to o^-) because at equilibrium, trade occurs at some price infinitely often. By assumed trading rule, trading price is the offer price when bid price is higher than offer price. Then by Definition 2 (iii), the agent offer to sell at price no higher than o^+ infinitely often almost surely. For $\omega \in Q_\lambda$, strategy $\hat{\sigma}$ specifies that beginning at T_λ, the agent should abstain from selling (that is, set $\sigma_{t4} = 0$) whenever σ would have set the offer price no higher than o^+. This abstention should continue until a date when either a cumulative value of ρ in sales has been forgone, or else η_t^σ would have been less than ρ.

Formally, for all $\omega \in \Omega$, let $\zeta^\sigma(\omega)$ denote the set of dates that the agent's offer price σ_{t3} is below o^+,

$$\zeta^\sigma(\omega) = \{t \,|\, \sigma_{t3}(\eta_0, \omega) < o^+\}.$$

The set $\zeta^\sigma(\omega)$ is infinite almost surely. Set $S = \rho$, $l_0 = T_\lambda$, and $n = 1$. For each $\omega \in Q_\lambda$, strategy σ is modified as follows.

1. First consider the offer strategy. For each $t > l_{n-1}$, if $t \in \zeta^\sigma(\omega)$, set $\hat{\sigma}_{t3} = \sigma_{t3}$ and $\hat{\sigma}_{t4} = \max\{0,\, \sigma_{t4} - S/\sigma_{t3}\}$, until the agent succeed selling $\hat{\sigma}_{t4}$ instead of σ_{t4} at date l_n,

$$l_n = \min\Big\{t \,|\, t > l_{n-1},\, t \in \zeta^\sigma(\omega),\, \omega_{t1} = b,\, \omega_{t2} \geq \sigma_{t3},\, \min\{\omega_{t3},\, \sigma_{t4}\} > \hat{\sigma}_{t4}\Big\}.$$

   Such a date exists because $\hat{\sigma}_{t4} < \sigma_{t4}$, and because by definition 2 (ii), the agent runs into buyers with bid as high as o^+ infinitely often. If $t \notin \zeta^\sigma(\omega)$, the agent behaves as if $(\rho - S)$ has not been spent at $l_1, \ldots, l_{n-1}$; $(\hat{\sigma}_{t3},\, \hat{\sigma}_{t4})(\eta_0 - (\rho - S), \omega) = (\sigma_{t3},\, \sigma_{t4})(\eta_0, \omega)$.

   For the bid strategy, for any $t > l_{n-1}$, there are two possibilities:

- At date t, the agent has enough money to purchase as he originally planned; i.e., $\eta_t^{\hat{\sigma}} = \eta_t^{\sigma} - (\rho - S) \geq \sigma_{t1}\sigma_{t2}$. In this case, again behave as if $(\rho - S)$ has not been spent at $l_1, \ldots, l_{n-1}$; $(\hat{\sigma}_{t1},\, \hat{\sigma}_{t2})(\eta_0 - (\rho - S), \omega) = (\sigma_{t1},\, \sigma_{t2})(\eta_0, \omega)$.
- It is discovered that $\omega \notin Q$. That is, after selling $\rho - S$ value less at $l_1, \ldots, l_{n-1}$, the agent does not have enough money to buy at t that he would have had he followed strategy σ, i.e., $\eta_t^{\hat{\sigma}} < \sigma_{t1}\sigma_{t2},$. In such a case, set $t_0 = l_{n-1}$ and go to step (3) directly.

For $l_{n-1} < t < l_n$, if the agent has enough money, trade is carried out as if he has not adopt the modified strategy $\hat{\sigma}$. Hence $v_t^{\hat{\sigma}} = v_t^{\sigma}$. At date l_n, the agent sells $\hat{\sigma}_{t4}$ units of his endowment instead of σ_{t4}, and consume the remaining portion of the endowment. Let y_n denote the reduction of the agent's sale at l_n,

$$y_n \equiv \min\{\omega_{l_n 3},\, \sigma_{l_n 4}\} - \min\{\omega_{l_n 3},\, \hat{\sigma}_{l_n 4}\}.$$

The utility gain is $v_{l_n}^{\hat{\sigma}} - v_{l_n}^{\sigma} = c\, y_n$, and the corresponding money loss is $\sigma_{l_n 3} y_n \leq (o^- + \delta) y_n$.

2. Let $S = S - \sigma_{l_n 3} y_n$. If $S > 0$, set $n = n + 1$, return to step (1). Otherwise, selling ρ-value less of endowment has been accomplished at time $l_{N(\omega)}$ where $N(\omega) = n$. $N(\omega)$ is finite because ρ is finite, and because trading quantities are determined by i.i.d. bid quantities and offer quantities which are non-binding constraint on reducing ρ to zero. Set $t_0 = l_n$, and go to step (3).

3. For all $t > t_0$, the agent resume strategy σ as if $(\rho - S)$ has not been spent previously, i.e., $\hat{\sigma}_t(\eta_0 - (\rho - S), \omega) = \sigma_t(\eta_0, \omega)$. When this is possible, $v_t^{\hat{\sigma}} = v_t^{\sigma}$. If the agent finds that he does not have enough money to buy at t what he would have had he followed strategy σ, then $\eta_t^{\sigma} < \rho$, or equivalently, $\omega \notin Q$. In such a case, $\hat{\sigma}$ prescribes that the agent buys what he can on dates when he can not afford to make the original bid. The number of such occasions is limited by the amount $\rho - S$ overspent previously. Formally, for all $t > t_0$, if $\eta_t^{\hat{\sigma}} < \sigma_{t1}\sigma_{t2}$, set $\hat{\sigma}_{t1} = \sigma_{t1}$, but $\hat{\sigma}_{t2} = \eta_t^{\hat{\sigma}} / \sigma_{t1}$. Let $R = \rho - S$ and $k = 1$. The following process records the utility loss as the agent buys less than under strategy σ.

   (a) Let t_k be the k-th time that the agent is unable to purchase what he originally planed under σ.

$$t_k = \min\Big\{ t \,|\, t > t_{k-1},\, \omega_{t1} = s,\, \omega_{t2} \leq \sigma_{t2}, \text{ and } \eta_t^{\hat{\sigma}} < \sigma_{t1}\sigma_{t2} \Big\}$$

Let z_k denote the reduction of the agent's purchase at t_k,

$$z_k \equiv \min\{\omega_{t_k 3},\, \sigma_{t_k 2}\} - \min\{\omega_{t_k 3},\, \hat{\sigma}_{t_k 2}\}.$$

Then the utility loss on date t_k is $v^{\sigma}_{t_k} - v^{\hat{\sigma}}_{t_k} = u\, z_k$, and the amount of money did not spend is $\omega_{t_k 1} z_k$.

(b) Let $R = R - \omega_{t_k 1} z_k$. If $R > 0$, set $k = k + 1$, and return to step (a). Otherwise, the money balance has been returned to what it would have been in the original strategy profile. The agent resumes strategy σ. Let $K(\omega) = k$.

As previously argued, this process ends in finite time.

To compare strategies $\hat{\sigma}$ and σ, let $D(\eta_0, \omega)$ denote the difference in utility generated by $\hat{\sigma}$ and σ for any $\omega \in \Omega$ and the given η_0, $D(\eta_0, \omega) = \sum_{\tau=0}^{\infty} \Big(v^{\hat{\sigma}}_{\tau}(\eta_0, \omega) - v^{\sigma}_{\tau}(\eta_0, \omega) \Big)$. Then

$$\begin{aligned} \forall\, \omega \in Q_\lambda \cap Q \qquad & D(\eta_0, \omega) = c \sum_{n=1}^{N(\omega)} y_n \geq \frac{c\,\rho}{o^+} \\ \forall\, \omega \in Q_\lambda \setminus Q \qquad & D(\eta_0, \omega) = c \sum_{n=1}^{N(\omega)} y_n - u \sum_{k=1}^{K(\omega)} z_k \geq -\frac{u\,\rho}{o^-}. \end{aligned} \tag{20}$$

By (19) and (20) and Fatou's Lemma (Royden[10]),[6]

$$\begin{aligned} \liminf_{t\to\infty} \mathsf{E}\Big[\sum_{\tau=0}^{t} \Big(v^{\hat{\sigma}}_{\tau}(\eta_0, \omega) - v^{\sigma}_{\tau}(\eta_0, \omega) \Big) \Big] &\geq \int \lim_{t\to\infty} \big(v^{\hat{\sigma}}_{\tau}(\eta_0, \omega) - v^{\sigma}_{\tau}(\eta_0, \omega) \big)\, dP(\omega) \\ &\geq \frac{c\,\rho}{o^+} P(Q_\lambda \cap Q) - \frac{u\,\rho}{o^-} P(Q_\lambda \setminus Q) \\ &\geq \frac{c\,\rho}{o^+}(1-\lambda) P(Q) - \frac{u\,\rho}{o^-} \lambda P(Q). \end{aligned} \tag{21}$$

Take $\lambda < c\,o^-/2(c\,o^- + u\,o^+)$, since $P(Q) \geq \varepsilon$, (21) implies

$$\begin{aligned} \liminf_{t\to\infty} \mathsf{E}\Big[\sum_{\tau=0}^{t} \Big(v^{\hat{\sigma}}_{\tau}(\eta_0, \omega) - v^{\sigma}_{\tau}(\eta_0, \omega) \Big) \Big] &\geq \Big(\frac{c\,\rho}{o^+}(1-\lambda) - \frac{u\,\rho}{o^-}\lambda \Big) P(Q) \\ &= \Big(\frac{c}{o^+} - \frac{c\,o^- + u\,o^+}{o^- o^+} \lambda \Big) \rho\, \varepsilon \\ &> \frac{c}{2\,o^+}\, \rho\, \varepsilon > 0. \end{aligned}$$

By (7), strategy $\hat{\sigma}$ overtakes strategy σ, which contradicts the assumption that σ is an equilibrium strategy. ∎

[6]The Fatou's Lemma stated on Royden[10], p86, requires the sequence that converges to be nonnegative. In fact, the lemma holds for sequence bounded from below, which is the case here.

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