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ABSTRACT

Recent research in international finance has equated changes in real exchange rates with differences between the marginal utility growths of representative agents in different economies. The asset market view of exchange rates, encapsulated in this equation, has been used to gain insights into exchange rate determination, foreign exchange risk premia, and international risk sharing. We argue that, in fact, this equation is of limited usefulness. By itself, the asset market view does not identify the economic mechanism that determines the exchange rate. It only holds under complete markets, and even then, it does not generally allow us to identify the marginal utility growths of distinct agents. Moreover, if we allow for incomplete asset markets, measures of agents' marginal utility growths, and international risk sharing, cannot be based on asset market and exchange rate data alone. Instead, we argue that in order to explain how exchange rates are determined, it is necessary to make specific assumptions about preferences, goods market frictions, the assets agents can trade, and the nature of endowments or production.

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Recent research in international finance has argued that changes in the real exchange rate reflect the difference in marginal utility growth, or stochastic discount factors (SDFs), between foreign and domestic investors. This *asset* market view of exchange rates, applied to representative agents in the U.S. and the U.K., is encapsulated in the simple equation:

$$\begin{array}{c}\text{growth in}\\ \text{real U.S./U.K.}\\ \text{exchange rate}\end{array} = \begin{array}{c}\text{U.K. representative agent's}\\ \text{marginal utility growth}\\ \text{over his consumption basket}\end{array} - \begin{array}{c}\text{U.S. representative agent's}\\ \text{marginal utility growth}\\ \text{over her consumption basket}\end{array}. \quad (1)$$

This equation has been used to draw inferences about international risk sharing, to characterize risk premia in the foreign exchange market, and to gain insights into exchange rate determination.[1] For example, Brandt, Cochrane, and Santa-Clara (2006) use Eq.(1), together with asset returns and real exchange rate volatility, to conclude that the marginal utility growth of American and British investors must be highly correlated. They argue that this high correlation in marginal utility growth implies a much higher degree of international risk sharing than does consumption data.

We argue that Eq. (1), by itself, is not useful for thinking about how real exchange rates are determined. Moreover, inference about risk sharing cannot rely solely on Eq. (1). Section 1 provides the basis for our argument. We show that Eq. (1) only needs to hold if asset markets are assumed to completely span all of the variation in agents' marginal utility growths. In this case, agents always equate marginal utility growth when it is measured in a common basket of goods (or other numeraire) that they can frictionlessly trade with each other. Thus, when Eq. (1) holds, there are only two possible reasons that marginal utility growth can differ across agents: either it is measured using different baskets of goods (i.e., different numeraires), or it is measured using a common basket of goods that has a different price in the two countries, due to frictions in the goods market. Both of these conditions are rooted in the *goods* market rather than the *asset* market.

For example, if American and British agents choose the same consumption basket, then the real exchange rate measures the relative price of this basket of goods across the two countries. To understand changes in the real exchange rate, in this situation, we must understand how goods markets frictions interact with the underlying forces in the economy, to create changes in the relative price of an identical basket of goods across the two countries. If, in addition, asset markets are complete, then any difference in the marginal utility growth

[1] See, to name just a few: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Smith and Wickens (2002); Brandt, Cochrane, and Santa-Clara (2006); Lustig and Verdelhan (2006); Brennan and Xia (2006); Lustig and Verdelhan (2007); Alvarez, Atkeson, and Kehoe (2007) Bakshi, Carr, and Wu (2008); Farhi, Fraiberger, Gabaix, Ranciere, and Verdelhan (2009); Verdelhan (2010); Colacito and Croce (2011); Lustig, Roussanov, and Verdelhan (2011); and Stathopoulos (2011).

of the two representative agents only reflects the change in the relative price of the same consumption basket across the two countries.

Alternatively, if American and British agents choose different consumption baskets, then the real exchange rate can vary because it represents the relative price of different baskets. Here, to understand real exchange rate behavior, we need to know something about how preferences over goods interact with changes in the relative abundance of the different goods in the baskets. Additionally, in this case, we would not want to draw any inference about risk sharing from variation in the real exchange rate. Even if asset markets are complete, so that Eq. (1) applies, marginal utility growths of the representative agents in the U.S. and U.K. can still differ because they are measured over different consumption baskets. In fact, when goods markets are frictionless, the growth in the exchange rate is always equal to *any* agent's (be they American, British, or otherwise) marginal utility growth measured over units of the U.K. consumption basket minus that *same* agent's marginal utility growth measured over units of the U.S. consumption basket.[2] This standard change of units (or numeraire) applies irrespective of asset market completeness.

More generally, when asset markets are incomplete Eq. (1) does not hold, which is an additional reason why imperfect risk sharing and real exchange rate variation are not one and the same thing. With incomplete markets, the right hand side of Eq. (1) is often reinterpreted as the difference between two SDFs (one for each country). Empirical SDFs are sometimes formed by projecting marginal utility growths in the two countries onto vectors of returns on the *same* assets, measured in the respective real currency units of the two countries. It is argued, for example, by Brandt, Cochrane, and Santa-Clara (2006) and Lustig and Verdelhan (2012), that Eq. (1) holds for these empirical minimum-variance SDFs, even under incomplete markets. In Section 1.2.1 we show that, in fact, Eq. (1) does *not* generally hold for these minimum-variance SDFs. Additionally, in Section 1.3 we make clear that minimum-variance SDFs provide a measure of shared risk, but are silent on the risks that agents do not share due to asset market incompleteness. In the language of finance, agents may have different marginal utility growths, however, the projection of every agent's marginal utility growth onto the space of asset returns must be identical. It is true that these projections may look different when they are measured in different units, but this result is only because they are identical when measured in common units.

We argue that models and assumptions are needed to explain variation in the real exchange rate. In Section 1.4 we describe the necessary ingredients in any such model. In

[2] Put differently, any agent's marginal utility growth measured over units of the U.K. consumption basket must equal his or her marginal utility growth measured over units of the U.S. consumption basket plus the growth in the number of units of the U.S. consumption basket that can be exchanged for one unit of the U.K. consumption basket.

Section 2 we consider a simple two-country endowment economy, closely based on Backus and Smith (1993). We assume that agents in these two economies have standard preferences over two goods. One good is frictionlessly traded. We alternatively assume that the second good is not traded, or is frictionlessly traded. In this model, real exchange rate *fluctuations* arise out of variation in countries' endowments of the two goods. But the exact nature of these fluctuations–the mapping from endowments to consumptions and real exchange rates–depends crucially on the precise nature of the assumed preference differences, imperfections in goods markets, and imperfections in asset markets.

Our model also shows that the link between risk sharing and real exchange rate variation is tenuous. We construct equilibria with lots of real exchange rate variation combined with perfect or no risk sharing. Similarly, we construct equilibria with no real exchange rate variation combined with perfect or no risk sharing. The weak connection between risk sharing and exchange rate behavior is readily understood if we consider a single economy with two agents, in which all goods have the same prices in all markets. The fact that these agents face the same prices and use the same numeraire does not tell us what the economy's overall risk sharing characteristics are. It only tells us that, if risk is not perfectly shared, goods market imperfections play no role. More generally, no model-free economic inferences about different agents can be made by only observing the prices of assets and goods that they trade at common prices, in a competitive equilibrium.

The literature sometimes uses reduced form models of the marginal utility growths of representative agents in different countries, in conjunction with the asset market view, to investigate the point-by-point behavior of exchange rates. In Section 3 we discuss problems with this approach. First, the reduced form model of the domestic agent's marginal utility growth is often identified using asset returns denominated in real domestic currency units, while the reduced form model for the foreign agent is identified using the *same* asset returns denominated, instead, in real foreign currency units. The real exchange rate is a necessary input into such exercises, so implications for the real exchange rate cannot be treated as an output as well. Moreover, typical models specify reduced forms that are functions of asset returns. If asset markets are not complete, such models cannot identify the marginal utility growths of representative agents because asset returns do not span the variation in marginal utility growths. Even if asset markets are complete, the list of returns used in the model must completely span variation in the two marginal utility growths. However, papers that in this literature only use a very small subset of the assets that are available for agents to invest in.

1 Real Exchange Rates

We begin with the standard definition of the real exchange rate between two agents. The agents could be located anywhere in the world, including places that use the same nominal currency to denominate prices (e.g., different countries within the Euro zone, or different states in the U.S.). For convenience, let Amy and Bob be the names of these two agents, and to be concrete, assume that they are the representative agents in the United States and the United Kingdom, respectively. We make the standard assumption that there is frictionless trade in nominal currencies so that, without loss of generality, we use U.S. dollars (or any other nominal currency) to denominate the prices of *all* goods and assets, regardless of their location. If the price of a good or asset is denominated in a different currency, then it can be converted to the U.S. dollar equivalent at the relevant nominal exchange rate.

Let P be the dollar price today in the U.S. of one unit of Amy's consumption basket of goods, and let P' be its dollar price next period. Similarly, let $\tilde{P}$ be the U.S. dollar price today in the U.K. of one unit of Bob's consumption basket, and let $\tilde{P}'$ be its dollar price next period. The real U.S./U.K. exchange rate, E, is defined as the relative value of a unit of Bob's consumption basket in the U.K. to a unit of Amy's consumption basket in the U.S.,

$$E \equiv \tilde{P}/P \quad \text{and} \quad E' \equiv \tilde{P}'/P' \,. \tag{2}$$

Empirically, P is usually measured as the dollar price of the basket of consumer goods and services that is used to compute the consumer price index (CPI) in the U.S. Likewise, let $\tilde{P}^*$ denote the U.K. pound price of the basket of consumer goods and services that is used to compute the CPI in the U.K. If S is the nominal dollar/pound exchange rate (i.e., the U.S. dollar price of one U.K. pound), then $\tilde{P} \equiv S\tilde{P}^*$ is the U.S. dollar price equivalent of this U.K. basket. Therefore, the usual measure of the *growth* in the real U.S./U.K. exchange rate is

$$\ln(E'/E) = \ln(\tilde{P}^{*\prime}/\tilde{P}^*) + \ln(S'/S) - \ln(P'/P) \,. \tag{3}$$

If the composition of Amy's and Bob's consumption baskets is the same *and* they face identical prices for the goods in their baskets, then the real U.S./U.K. exchange rate is constant. Therefore, the real U.S./U.K. exchange rate can only vary if either (or both):

1. The composition of Amy's and Bob's consumption baskets is different; *or*

2. Amy and Bob face different prices, in the U.S. and the U.K., for identical goods in their baskets (where prices are measured in common units such as U.S. dollars).

Thus, to understand *why* the real exchange rates varies, at a minimum it is necessary to understand *why* agents may face different prices for identical goods and/or *why* the composition of their consumption baskets may differ.

From an empirical standpoint, *both* of these conditions for a variable real exchange rate are satisfied for virtually every country pair in the world. Different countries use different baskets of consumer goods and services to compute their respective consumer price indices. Also, identical goods frequently have different prices in different countries (i.e., purchasing power parity does not typically hold across countries, or even in different locations within the same country).

1.1 The Asset Market View of Real Exchange Rates

As its name suggests, the asset market (or SDF) view of real exchange rates focuses on the role asset markets play in determining real exchange rates. Consider a set of k assets that can be located anywhere in the world. The standard assumption in this literature is that trade in assets is frictionless, so that Amy and Bob can both trade the same set of k assets at the same prices. Let $\mathbf{X}$ be the $k \times 1$ vector of uncertain asset payoffs next period, and let $\mathbf{P_X}$ be the $k \times 1$ vector of asset prices today. Again, without loss of generality, we assume that the payoffs and prices of all the assets are measured in U.S. dollars.

Let λ be Amy's marginal utility today from an additional unit of her consumption basket, and let λ' denote the (uncertain) discounted value of her marginal utility next period. Note that Amy's indirect marginal utility today of an additional dollar is λ/P, since she can purchase $1/P$ units of her consumption basket with that dollar. If Amy maximizes her expected discounted utility, then her first order condition for optimality (i.e., her Euler equation) implies that

$$\mathbf{P_X}\frac{\lambda}{P} = \mathbb{E}\left[\mathbf{X}\frac{\lambda'}{P'}\right] \,, \quad \text{or equivalently,} \quad \mathbf{1} = \mathbb{E}\left[\mathbf{R}\frac{P}{P'}\frac{\lambda'}{\lambda}\right] \,. \tag{4}$$

In Eq. (4), $\mathbf{1}$ denotes a $k \times 1$ vector of 1's, $\mathbf{R} \equiv \mathbf{X}/\mathbf{P_X}$ denotes the $k \times 1$ vector of dollar-denominated asset returns (using element-by-element division), and $\mathbb{E}[\cdot]$ denotes the standard expectations operator. Euler equations are typically used in financial economics to understand cross-sectional variation in the *average* (or *expected*) returns of different assets, based on how those returns covary with the marginal utility growth of agents.[3]

Analogous to Amy, let $\tilde{\lambda}$ denote Bob's marginal utility today over a unit of his consumption basket and let $\tilde{\lambda}'$ denote the (uncertain) discounted value of his marginal utility next

[3]Note that Euler equations are only first order conditions and therefore are *not* typically used to understand the *realized* asset returns in *every* period.

period. If Bob also maximizes his expected discounted utility, then his first order condition for optimality implies that

$$\mathbf{1} = \mathbb{E}\left[\mathbf{R}\frac{\tilde{P}}{\tilde{P}'}\frac{\tilde{\lambda}'}{\tilde{\lambda}}\right] . \tag{5}$$

If the asset returns in Eqs. (4) and (5) completely span all of the variation in Amy's and Bob's indirect marginal utility growths, then they must *always* (i.e., in *every* state of the world next period) equate indirect marginal utility growth, so that

$$\frac{P}{P'}\frac{\lambda'}{\lambda} = \frac{\tilde{P}}{\tilde{P}'}\frac{\tilde{\lambda}'}{\tilde{\lambda}} . \tag{6}$$

Eq. (6) can be rearranged and written in logs to produce the asset market view of exchange rates in Eq. (1),

$$\underbrace{\begin{array}{c}\text{growth in}\\ \text{real U.S./U.K.}\\ \text{exchange rate}\end{array}}_{\ln E' - \ln E} = \underbrace{\begin{array}{c}\text{U.K. representative agent's}\\ \text{marginal utility growth over}\\ \text{his consumption basket}\end{array}}_{\ln \tilde{\lambda}' - \ln \tilde{\lambda}} - \underbrace{\begin{array}{c}\text{U.S. representative agent's}\\ \text{marginal utility growth over}\\ \text{her consumption basket}\end{array}}_{\ln \lambda' - \ln \lambda} , \tag{7}$$

where the real exchange rate is defined by Eq. (2). To be clear, the asset market view of exchange rates in Eq. (7) says that Bob's consumption basket *always* (i.e., in *every* state/period) appreciates in value relative to Amy's consumption basket whenever his marginal utility growth is larger than hers, and it *always* depreciates in value when the opposite is true. In other words, the real U.K. pound *always* appreciates against the real U.S. dollar whenever times become "worse" in the U.K. relative to the U.S., and vice versa.

The recent literature in international asset pricing has made extensive use of the asset market view of exchange rates. For example, Brandt, Cochrane, and Santa-Clara (2006) use Eq. (7) and the volatility of real exchange rates to make inferences about the extent of international risk sharing.[4] Lustig and Verdelhan (2007) argue that the risks of foreign currency investments are *determined by* differences in agents' marginal utility growths.[5]

[4] We provide a detailed analysis of Brandt, Cochrane, and Santa-Clara (2006) in Section 1.2.

[5] Section IIIB ("Mechanism: Where Do Consumption Betas of Currencies Come From?") of Lustig and Verdelhan (2007) states that:

> Investing in foreign currency is like betting on the difference between your own and your neighbor's intertemporal marginal rate of substitution (IMRS). These bets are very risky if your IMRS is not correlated with that of your neighbor, but they provide a hedge when her IMRS is highly correlated and more volatile.

Likewise, Verdelhan (2010) argues that changes in the real exchange rate are *determined by* differences in the marginal utility growths of foreign and domestic investors.[6] Other examples of papers that rely on the asset market view of exchanges rates are cited in the first footnote of the introduction.

Our main point in this paper is that, alone, the asset market view of exchange rates is not helpful for understanding variation in the real exchange rate. The intuition for our main point is simple. The real exchange rate can only vary if either (or both) the composition of Amy's and Bob's consumption baskets differs, or they face different prices for identical goods in their baskets. Therefore, as we stated earlier, to understand *why* the real exchange rates varies, at a minimum it is necessary to understand *why* agents may face different prices for identical goods and/or *why* the composition of their consumption baskets may differ. However, the asset market view of exchange rates provides no economic insights on either front. In fact, it *does not* even require a stand on the specific nature of goods markets, but instead relies only on complete asset markets. Indeed, the broad appeal of the asset market view stems from the fact that it *only* relies on asset markets.[7]

The following example explicitly illustrates that complete asset markets is not a sufficient condition to understand, or give an economic interpretation to, variation in the real exchange rate. Suppose that Amy and Bob face the same prices in the U.S. and the U.K. for identical goods in their baskets. In this case, Amy can trade one unit of Bob's basket in the U.S. for $\tilde{P}/P \equiv E$ units of her basket. Therefore, her marginal utility from an additional unit of Bob's basket is λE. Likewise, Bob's marginal utility from an additional unit of Amy's basket is $\tilde{\lambda}/E$ (since, in the U.K., he can trade a unit of Amy's basket for $P/\tilde{P} \equiv 1/E$ units of his basket). Thus, if Amy and Bob face the same prices for identical goods, then they are *not uniquely identified* in the asset market view of real exchange rates, since an equivalent

[6] The introduction of Verdelhan (2010) states that:

> When markets are complete, the real exchange rate, measured in units of domestic goods per foreign good, equals the ratio of foreign to domestic pricing kernels. Exchange rates thus depend on foreign and consumption growth shocks. If the conditional variance of the domestic stochastic discount factor (SDF) is large relative to its foreign counterpart, then domestic consumption growth shocks determine variation in exchange rates.

[7] For example, in their introduction, Brandt, Cochrane, and Santa-Clara (2006) state:

> Yet the conclusion (that international risk sharing is better than you think) is hard to escape. Our calculation uses only price data, and no quantity data or economic modeling (utility functions, income or productivity shock processes, and so forth). A large degree of international risk sharing is an inescapable logical conclusion.

economic interpretation of Eq. (7) is

$$\underbrace{\begin{array}{c}\text{growth in}\\\text{real U.S./U.K.}\\\text{exchange rate}\end{array}}_{\ln E'-\ln E} = \underbrace{\begin{array}{c}\text{U.S. representative agent's}\\\text{marginal utility growth over}\\\text{U.K. consumption basket}\end{array}}_{\ln(\lambda' E')-\ln(\lambda E)} - \underbrace{\begin{array}{c}\text{U.K. representative agent's}\\\text{marginal utility growth over}\\\text{U.S. consumption basket}\end{array}}_{\ln(\tilde{\lambda}'/E')-\ln(\tilde{\lambda}/E)} . \tag{8}$$

Therefore, when agents face the same prices for identical goods, it is impossible to infer any differences in their marginal utility growths using only the growth in the real exchange rate. Conversely, since the agents are not uniquely identified when they face the same goods prices, growth in the real exchange cannot possibly be *determined by* differences in their marginal utility growths.

Note that the math below the braces in Eq. (8) always holds whenever Eq. (6) holds. However, the economic *interpretation* provided above the braces in Eq. (8) is only valid if the representative agents in the U.S. and the U.K. face the same prices for identical goods. It is also worth noting that, if agents face the same prices for identical goods, then a version of Eq. (7) holds for any *single* agent, regardless of whether asset markets are complete or incomplete. For example, we can write

$$\underbrace{\begin{array}{c}\text{growth in}\\\text{real U.S./U.K.}\\\text{exchange rate}\end{array}}_{\ln E'-\ln E} = \underbrace{\begin{array}{c}\text{U.S. representative agent's}\\\text{marginal utility growth over}\\\text{U.K. consumption basket}\end{array}}_{\ln(\lambda' E')-\ln(\lambda E)} - \underbrace{\begin{array}{c}\text{U.S. representative agent's}\\\text{marginal utility growth over}\\\text{U.S. consumption basket}\end{array}}_{\ln \lambda'-\ln \lambda} . \tag{9}$$

Again, the math below the braces in Eq. (9) holds trivially, but the economic interpretation requires that agents face the same prices for identical goods.

Eq. (8) emphasizes two important points. First, Amy and Bob are not uniquely identified in the asset market view of real exchange rates unless they face *different* prices for identical goods. In other words, complete asset markets is *not* a sufficient condition to provide a unique economic interpretation to the asset market view of real exchange rates in Eq. (7). At a minimum, it is also necessary to assume that the representative agents face different prices for identical goods. Second, if the composition of Amy's and Bob's consumption baskets *differs* (which, in empirical implementations, is virtually always the case), then at least a portion of any variation in the real U.S./U.K. exchange rate could simply reflect changes in the relative price of different baskets of goods. At the extreme, if Amy and Bob face identical prices for all goods, then, as Eq. (7) and (8) illustrate, *all* of the variation in the real exchange rate is accounted for by changes in the relative price of different baskets.

Thus, to understand *why* the real exchange rate varies, at a minimum it is necessary to understand *why* agents may face different prices for identical goods *and* correctly account for any differences in the composition of their consumption baskets (ideally, one would also endeavor to understand *why* the composition of their consumption baskets differ).

1.2 Inferences About International Risk Sharing

In this section we provide a more detailed analysis of Brandt, Cochrane, and Santa-Clara (2006). As we mentioned in Section 1.1, they use the asset market view of exchange rates in Eq. (7) to interpret variation in the real exchange rate as evidence that foreign and domestic investors have not shared risk perfectly. To measure the *extent* of imperfect risk sharing, they take the variance of both sides of Eq. (7). From the abstract of Brandt, Cochrane, and Santa-Clara (2006):

> Exchange rates depreciate by the difference between domestic and foreign marginal utility growth or discount factors. Exchange rates vary a lot, as much as 15% per year. However, equity premia imply that marginal utility growth varies much more, by at least 50% per year. Therefore, marginal utility growth must be highly correlated across countries: international risk sharing is better than you think. Conversely, if risks really are not shared internationally, exchange rates should vary more than they do: exchange rates are too smooth.

What exactly constitutes perfect risk sharing, and how is it related to the difference in agents' marginal utility growths?

First, consider again the example in Section 1.1 in which Amy and Bob face the same prices in the U.S. and the U.K. for identical goods in their baskets. In that example we showed that Amy's marginal utility from an additional unit of Bob's basket is λE, while Bob's marginal utility from an additional unit of Amy's basket is $\tilde{\lambda}/E$. Therefore, if asset markets are complete and agents face the same prices for identical goods in their baskets then, in *every* period,

$$\underbrace{\begin{array}{c}\text{U.S. representative agent's}\\ \text{marginal utility growth over}\\ \text{U.S. consumption basket}\end{array}}_{\ln\lambda'-\ln\lambda} = \underbrace{\begin{array}{c}\text{U.K. representative agent's}\\ \text{marginal utility growth over}\\ \text{U.S. consumption basket}\end{array}}_{\ln(\tilde{\lambda}'/E')-\ln(\tilde{\lambda}/E)}, \tag{10}$$

and

$$\underbrace{\begin{array}{c}\text{U.S. representative agent's}\\ \text{marginal utility growth over}\\ \text{U.K. consumption basket}\end{array}}_{\ln(\lambda' E')-\ln(\lambda E)} = \underbrace{\begin{array}{c}\text{U.K. representative agent's}\\ \text{marginal utility growth over}\\ \text{U.K. consumption basket}\end{array}}_{\ln\tilde{\lambda}'-\ln\tilde{\lambda}}. \tag{11}$$

It is straightforward to extend this analysis to show that, when asset markets are complete, Amy and Bob *always* equate marginal utility growth over *any* basket of goods for which they face identical prices. In particular, if Amy and Bob face identical prices for *all* goods (and services), then with complete asset markets they *always* (i.e., in *every* state/period) equate marginal utility growth over *any* common basket of goods (and services). In this case, Amy and Bob share risk perfectly. However, note that even if Amy and Bob share risk perfectly, their marginal utility growths in Eq. (7) can still differ when they are measured over *different* baskets of goods (and services). In this case, the real exchange rate can also still vary because it too reflects the relative price of *different* baskets.

With frictionless trade in assets, risk sharing can only be imperfect if asset markets are incomplete *and/or* agents face different prices for identical goods. If asset markets are incomplete, then Amy's and Bob's marginal utility growths can differ across states of the world that are not spanned by the available assets. If they face different prices for identical goods, then there must be a friction in the goods market that prevents these prices from being equal in different locations, and that friction may also prevent perfect risk sharing.

Contrary to the premise of Brandt, Cochrane, and Santa-Clara (2006), the conditions for imperfect risk sharing *do not* completely overlap with the conditions for a variable real exchange rate. For example, as we noted earlier, if the composition of Amy's and Bob's consumption baskets is the same *and* they face identical prices for the goods in their baskets, then the real exchange rate is constant. Yet, risk sharing can still be imperfect if asset markets are incomplete. Conversely, if the asset market is complete and agents face identical prices for the goods in their baskets, then risk sharing is perfect (since they *always* equate marginal utility growth over *any* common basket of goods). Yet the exchange rate can still vary if the composition of Amy's and Bob's consumption baskets differs.

Table 1 provides the necessary conditions such that variation in the real exchange rate is a direct reflection of imperfect risk sharing. When asset markets are complete and the composition of Amy's and Bob's consumption baskets is identical, then the exchange rate can only vary if they face different prices for identical goods in their baskets. Likewise, if asset markets are complete, then risk sharing can only be imperfect if there are goods

	Composition of Consumption Baskets	
Asset Markets	Identical	Different
Complete	Yes	No
Incomplete	No	No

Table 1: Does a variable real exchange rate directly reflect imperfect risk sharing?

market frictions that prevent equal prices for identical goods. In other words, under these specific assumptions, both imperfect risk sharing and variation in the real exchange rate can only occur if there are frictions in the goods market. Unfortunately, as Table 1 indicates, the link between risk sharing and the volatility of the real exchange rate only holds in this one special case. As previously noted, the difference in composition of Amy's and Bob's consumption baskets can contribute to variation in the real exchange rate, but it need not affect risk sharing. Likewise, incomplete asset markets can contribute to imperfect risk sharing, without affecting the volatility of the real exchange rate.

To emphasize, if we only observe variation in the real exchange rate, and nothing more, then we cannot be sure of the extent to which that variation reflects goods market frictions versus different baskets of goods. Similarly, incomplete asset markets are a source of imperfect risk sharing, but one cannot learn the *extent* of market incompleteness (i.e., the extent to which agents' marginal utility growths are not spanned by asset returns) from asset returns alone. Therefore, very specific *assumptions* are required to make any inferences about international risk sharing using only observations of asset returns and variation in the real exchange rate. Moreover, any such inferences are not robust to different assumptions.

1.2.1 International Risk Sharing with Incomplete Asset Markets

Brandt, Cochrane, and Santa-Clara (2006) also examine international risk sharing when asset markets do not completely span agents' marginal utility growths. As the basis for their analysis, they claim that, even when asset markets are incomplete, Eq. (7) continues to hold for the *linear projection*s of agents' marginal utility growths onto the available asset returns (denominated in units of their respective consumption baskets).[8] Here, we show that Eq. (7) *does not* in fact hold for these linear projections.

[8]On page 675 of Brandt, Cochrane, and Santa-Clara (2006) they state that:

> These discount factors are the projections of any possible domestic and foreign discount factors onto the relevant spaces of asset payoffs, and they are also the minimum-variance discount factors. We show that Eq. (1) continues to hold with this particular choice of discount factors.

In Section 14.2.2 of Lustig and Verdelhan (2012) they also claim that Eq. (13) holds with incomplete markets. Alvarez, Atkeson, and Kehoe (2007) also make this claim.

Let $m \equiv \lambda'/\lambda$ and $\tilde{m} \equiv \tilde{\lambda}'/\tilde{\lambda}$ denote the discounted marginal utility growths of, respectively, Amy and Bob. Let $\mathbf{r} \equiv \mathbf{R}P/P'$ and $\tilde{\mathbf{r}} \equiv \mathbf{R}\tilde{P}/\tilde{P}'$ be the asset returns denominated in units of their respective consumption baskets. Let

$$\text{proj}\,[m\,|\mathbf{r}] \equiv \boldsymbol{\beta} \cdot \mathbf{r} \quad \text{and} \quad \text{proj}\,[\tilde{m}\,|\tilde{\mathbf{r}}] \equiv \tilde{\boldsymbol{\beta}} \cdot \tilde{\mathbf{r}} \tag{12}$$

denote the linear projections of Amy's and Bob's marginal utility growths onto, respectively, $\mathbf{r}$ and $\tilde{\mathbf{r}}$. In Eq. (12), $\boldsymbol{\beta}$ and $\tilde{\boldsymbol{\beta}}$ are $k \times 1$ vectors, and $\cdot$ denotes the dot product of two vectors. Brandt, Cochrane, and Santa-Clara (2006) claim that the asset market view of exchange rates in Eq. (7) continues to hold for these linear projections, so that

$$\ln E' - \ln E = \ln[\tilde{\boldsymbol{\beta}} \cdot \tilde{\mathbf{r}}] - \ln[\boldsymbol{\beta} \cdot \mathbf{r}] \quad \text{or equivalently,} \quad \tilde{\boldsymbol{\beta}} \cdot \tilde{\mathbf{r}} = \boldsymbol{\beta} \cdot \mathbf{r}\,(E'/E)\ . \tag{13}$$

Eq. (13) *does not* in fact hold if the number of assets used in the projections is less than the number of possible states of the world. Since, by definition, $\mathbf{r} \equiv \tilde{\mathbf{r}}E'/E$ notice that Eq. (13) implies

$$\tilde{\boldsymbol{\beta}} \cdot \tilde{\mathbf{r}} = \boldsymbol{\beta} \cdot \tilde{\mathbf{r}}\,(E'/E)^2\ . \tag{14}$$

However, in general, $\tilde{\mathbf{r}}\,(E'/E)^2$ is not in the linear span of $\tilde{\mathbf{r}}$ unless there are as many assets as future possible states of the world. To be more explicit, suppose that there are n states of the world next period indexed by z, and let $\tilde{\mathbf{r}}\,(z)$ and $E'\,(z)$ denote the values of $\tilde{\mathbf{r}}$ and E' in state z. Then Eq. (14) can be written more formally as

$$\tilde{\boldsymbol{\beta}} \cdot \tilde{\mathbf{r}}\,(z) = \boldsymbol{\beta} \cdot \tilde{\mathbf{r}}\,(z)\,[E'\,(z)\,/E]^2\ , \quad z = 1, \ldots, n\,. \tag{15}$$

Take any possible value of $\tilde{\boldsymbol{\beta}}$ as given. If the number of securities, k, is less than the number of states of the world next period, n, then Eq. (15) represents n equations in the k unknown elements of $\boldsymbol{\beta}$. Therefore, in general, we cannot find vectors $\boldsymbol{\beta}$ and $\tilde{\boldsymbol{\beta}}$ such that Eq. (15) holds. As a consequence, the same can be said of Eq. (13).

1.3 Projections onto Asset Returns

In the previous section we showed that Eq. (7) *does not* hold for the *linear projection*s of agents' marginal utility growths onto the asset returns (denominated in units of their respective consumption baskets). In this section we consider whether it is possible to learn anything about risk sharing using only the projections of agents' marginal utility growths onto the returns of assets that they can frictionlessly trade with each other.

Let $M \equiv (\lambda'/P')/(\lambda/P)$ and $\tilde{M} \equiv (\tilde{\lambda}'/\tilde{P}')/(\tilde{\lambda}/\tilde{P})$ denote the discounted indirect marginal utility growths of, respectively, Amy and Bob, defined over U.S. dollars. Let

$$\text{proj}\,[M\,|\mathbf{R}] \equiv \mathbf{R}\cdot\boldsymbol{\alpha} \quad \text{and} \quad \text{proj}[\tilde{M}\,|\mathbf{R}] \equiv \mathbf{R}\cdot\tilde{\boldsymbol{\alpha}} \tag{16}$$

denote the linear projections of Amy's and Bob's indirect marginal utility growth over dollars onto the space of available dollar-denominated asset returns, so that

$$M = \mathbf{R}\cdot\boldsymbol{\alpha} + \varepsilon\,, \quad \text{where} \quad \mathbb{E}\,[\mathbf{R}\varepsilon] = \mathbf{0}\,, \tag{17}$$

and

$$\tilde{M} = \mathbf{R}\cdot\tilde{\boldsymbol{\alpha}} + \tilde{\varepsilon}\,, \quad \text{where} \quad \mathbb{E}\,[\mathbf{R}\tilde{\varepsilon}] = \mathbf{0}\,. \tag{18}$$

It is well known that if Amy's and Bob's Euler equations (Eqs. 4 and 5) are both satisfied, then the linear projections in Eqs. (17) and (18) must agree, so that

$$\mathbb{E}\,[\mathbf{R}M] = \mathbf{1} = \mathbb{E}[\mathbf{R}\tilde{M}] \quad \Rightarrow \quad \boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}}\,. \tag{19}$$

From these projections we learn that Amy's and Bob's indirect marginal utility growths share a common component, but we learn nothing about the components of their indirect marginal utility growths, ε and $\tilde{\varepsilon}$, that are orthogonal to the available asset returns. These include all of Amy's and Bob's risks that go unshared due to asset market incompleteness, as well as any shared risk that happens to not be spanned by the available asset returns (more generally, shared risk not spanned by the returns used in the projections). Simply put, in equilibrium Amy and Bob must agree on the prices of assets and goods that they can frictionlessly trade with each other, and therefore it is impossible to learn where their marginal utility growths differ using only these prices.

1.4 Necessary Ingredients in a Model

What are the minimum ingredients required by a model that attempts to explain real exchange rates? As we've already argued, to understand *why* the real exchange rates varies, at a minimum it is necessary to understand *why* agents may face different prices for identical goods and/or *why* the composition of their consumption baskets may differ (or at least correctly account for any differences in the composition of their consumption baskets). Therefore, any model that attempts to explain how real exchange rates are *determined* must directly specify:

- agents' preferences over goods;

- any frictions in goods market trade that may restrict prices of identical goods from being equal in different locations;
- agents' preferences over time and uncertainty;
- the available assets that agents can trade; and
- endowments or production technology.

These ingredients are required to map structural shocks to endowments (or production technology) into agents' consumption and, hence, outcomes of the real exchange rate. Any model that is missing one of these key ingredients is silent about how the real exchange rate is determined.

A number of recent papers omit some of these necessary ingredients, but instead purport to explain variation in the real exchange rate using consumption-based asset pricing models together with the asset market view in Eq. (7). These papers *start* with aggregate consumption in the foreign and domestic economy and assume a utility function for the representative agent in each economy. For example, Verdelhan (2010) uses representative agents with external habit preferences (see Campbell and Cochrane, 1999), while Bansal and Shaliastovich (2010) uses representative agents with Epstein-Zin recursive utility functions (see Epstein and Zin, 1989).

As we noted at the beginning of Section 1, the real exchange rate is the relative price of a unit of the foreign representative agent's consumption basket to a unit of the domestic representative agent's basket. In equilibrium, prices (e.g., the real exchange rate) and quantities (e.g., aggregate consumption in the foreign and domestic economies) are *jointly* determined. Therefore, aggregate consumption cannot be the starting point in any model that endeavors to understand realized changes in the real exchange rate in *every* period.[9] Instead, the starting point must be agents' endowments (or production), and the model must provide map between those endowments and their ultimate consumption baskets.[10] A model

[9]Of course there *may* be asset markets, endowments (production technology), goods market frictions, and preferences over goods that *could* generate these equilibrium aggregate consumptions of the representative agents. Our point is that these elements are required to understand how aggregate consumption and the real exchange rate are *jointly* determined, but they are missing in many (if not most) of the recent papers in this literature.

[10]Aggregate consumption and the aggregate endowment (production) must be equal in a closed economy without physical investment or government purchases, but they can differ in an open economy. Colacito and Croce (2011) start with agents' endowments. However, they assume complete home bias, so that there is no trade in the goods market or the asset market, and therefore an agent's consumption is the same as his or her endowment. Unfortunately, with complete home bias, the real exchange rate is not uniquely determined. *Any* real exchange rate clears the market since agents are assumed to have no desire to consume goods that they are not endowed with. Nevertheless, Colacito and Croce (2011) use the asset market view of real exchange rates in Eq. (7) to uniquely characterize the growth in the real exchange rate.

that *starts* with aggregate consumption is silent about this map and, hence, is silent on the mechanism that determines the real exchange rate.

It is worth emphasizing that standard asset pricing models that start with aggregate consumption *can* be useful for understanding cross-sectional variation in the *average* (or *expected*) returns of different assets (including investments in nominal currencies) based on how those returns covary with the marginal utility growth of agents. Our point is simply that these models are *not* helpful for understanding realized changes in the real exchange rate *each period* (rather than average, or expected, changes).

Any model that purports to explain a real exchange rate must pass two simple diagnostics. First, the model cannot rely on complete asset markets to *define* the real exchange rate. The real exchange rate is the relative price of one basket of goods to another, and that relative price does not require asset markets to be complete. Therefore, any model must describe how the real exchange rate is determined, regardless of whether asset markets are complete or incomplete. Second, the *level* of the real exchange rate should be determined in the model (up to a constant that reflects the size of the baskets of goods that are being compared), and not simply *growth* (or changes) in the real exchange rate. If a model doesn't help us to understand the relative price of one basket of goods to another, then it doesn't help us to understand how the real exchange rate is determined. Models of the real exchange rate that rely *solely* on the asset market view in exchange rates in Eq. (7) fail both of these diagnostics.

2 An Endowment Economy

In Section 1 we showed that the asset market view of exchange rates in Eq. (7) is a first order condition that must be satisfied in any model, but it is not a substitute for a model. Marginal utility (aggregate consumption) growth does not determine the real exchange rate any more than the real exchange rate determines marginal utility (aggregate consumption) growth. A full-fledged model with the minimum necessary ingredients described in Section 1.4 is required to understand how aggregate consumptions and the real exchange rate are *jointly* determined in equilibrium. In this section we provide an example of such a model. The model is a generalization of Backus and Smith (1993) along three dimensions. First, we allow for preference differences across countries. Second, we allow for incomplete markets. Third, like Backus and Smith (1993) we have two goods, but we explicitly compare the case where the second good is nontraded to the case in which it is frictionlessly traded. We don't view this model as a solution to existing exchange rate puzzles, rather it is illustrative of our point about the joint determination of consumption and the real exchange rate.[11]

We describe an endowment economy with two countries ("home" and "foreign") and representative households within each country. Utility is defined over two goods, A and B. All goods are perishable and households live for two periods.

The representative household in the home economy has an instantaneous utility function

$$U(c_A, c_B) = u[c(c_A, c_B)], \tag{20}$$

where c_A and c_B denote, respectively, the consumption of goods A and B by the home household, $c(\cdot)$ is a homogeneous of degree 1 quasi-concave function of its arguments, and u is a monotonic function with standard properties. Similarly, the representative household in the foreign economy has the instantaneous utility function

$$\tilde{U}(\tilde{c}_A, \tilde{c}_B) = u[\tilde{c}(\tilde{c}_A, \tilde{c}_B)], \tag{21}$$

where $\tilde{c}_A$ and $\tilde{c}_B$ denote, respectively, the consumption of goods A and B by the foreign household, and $\tilde{c}(\cdot)$ is a homogeneous of degree 1 quasi-concave function of its arguments.

Both economies are cashless and use good A as the numeraire. Our model would have the same implications for the real exchange rate if we chose different numeraires. Goods markets meet sequentially. Good A is frictionlessly tradable. We alternately assume that good B is frictionlessly tradable or nontradable. We let P_B and P'_B denote the prices of good B in the home economy in the first and second periods. Similarly, we let $\tilde{P}_B$ and $\tilde{P}'_B$ denote the prices of good B in the foreign economy in the first and second periods. When good B is frictionlessly tradable, its price must be the same in both countries,

$$P_B = \tilde{P}_B \quad \text{and} \quad P'_B = \tilde{P}'_B\,. \tag{22}$$

The natural definition of the consumer price index (CPI) in the home country is a variable P such that $c_A + P_B c_B = P\, c(c_A, c_B)$. Since $c(\cdot)$ and $\tilde{c}(\cdot)$ are homogeneous of degree one functions, it can be shown that there are homogeneous of degree one functions $H(\cdot)$ and

[11]Dating back, at least, to the contributions of Stockman (1980) and Lucas (1982), a large literature has developed explicit equilibrium models of the exchange rate. To name but a few, Cole and Obstfeld (1991), Backus, Kehoe, and Kydland (1992), Dumas (1992), Backus and Smith (1993), Baxter and Crucini (1995), Obstfeld and Rogoff (1995), Sercu, Uppal, and Van Hulle (1995), Stockman and Tesar (1995), Betts and Devereux (1996), Chari, Kehoe, and McGrattan (2002), Apte, Sercu, and Uppal (2004), Bacchetta and Wincoop (2006), Kocherlakota and Pistaferri (2007), Pavlova and Rigobon (2007), Benigno and Thoenissen (2008), and Corsetti, Dedola, and Leduc (2008).

$\tilde{H}(\cdot)$ whose form depends on $c(\cdot)$ and $\tilde{c}(\cdot)$, such that the home and foreign CPIs are:[12]

$$P = H(1, P_B) \quad \text{and} \quad \tilde{P} = \tilde{H}(1, \tilde{P}_B)\,. \tag{23}$$

Similarly the CPIs in period two are

$$P = H(1, P'_B) \quad \text{and} \quad \tilde{P}' = \tilde{H}(1, \tilde{P}'_B)\,. \tag{24}$$

Identical to Eq. (2) in Section 1, the real exchange rates in periods one and two are

$$E \equiv \tilde{P}/P \quad \text{and} \quad E' \equiv \tilde{P}'/P'. \tag{25}$$

In the special case where preferences are identical in the two countries, we have $H(\cdot) = \tilde{H}(\cdot)$. If, additionally, both goods are traded, $E = 1 = E'$, regardless of the asset market structure. If preferences differ across countries and both goods are traded, variation in the real exchange rate *can* arise even though $\tilde{P}_B = P_B$. All that is needed is variation in P_B. We can make these statements even though we've said nothing about asset markets. This is one concrete sense in which the link between exchange rates and asset markets is tenuous.

As was the case in Section 1, we assume that there are k assets with $k \times 1$ random payoff vector $\mathbf{X}$. The $k \times 1$ price vector today for these assets is $\mathbf{P_X}$. The payoffs and prices of the assets are measured in units of good A. The asset payoffs, and all variables in period two, depend on the state of the world in period two. For notational simplicity, however, we suppress the dependence of period two variables on the state of the world.

The household in the home country chooses c_A, c_B, c'_A, c'_B, and the $k \times 1$ vector $\mathbf{a}$, to maximize

$$u[c(c_A, c_B)] + \beta\, \mathbb{E}\left\{u[c(c'_A, c'_B)\right\}\,, \tag{26}$$

subject to

$$c_A + P_B c_B + \mathbf{P_X} \cdot \mathbf{a} = y_A + P_B y_B \quad \text{and} \quad c'_A + P'_B c'_B = y'_A + P'_B y'_B + \mathbf{X} \cdot \mathbf{a}\,. \tag{27}$$

Here $0 < \beta < 1$, c_A and c_B are the household's current consumption of the two goods, c'_A and c'_B are the household's plans for future consumption of the two goods (in every possible state of the world), the jth element of $\mathbf{a}$ is the household's net purchases of asset j, and y_A, y_B, y'_A and y'_B are the household's current and future endowments of the two goods.

[12] For details, see the section on price aggregation in the appendix.

Similarly, the foreign household chooses $\tilde{c}_A$, $\tilde{c}_B$, $\tilde{c}'_A$, $\tilde{c}'_B$, and $\tilde{\mathbf{a}}$ to maximize

$$u[\tilde{c}(\tilde{c}_A, \tilde{c}_B)] + \beta\, \mathbb{E}\left\{u[\tilde{c}(\tilde{c}'_A, \tilde{c}'_B)]\right\} , \tag{28}$$

subject to

$$\tilde{c}_A + \tilde{P}_B \tilde{c}_B + \mathbf{P_X} \cdot \tilde{\mathbf{a}} = \tilde{y}_A + \tilde{P}_B \tilde{y}_B \quad \text{and} \quad \tilde{c}'_A + \tilde{P}'_B \tilde{c}'_B = \tilde{y}'_A + \tilde{P}'_B \tilde{y}'_B + \mathbf{X} \cdot \tilde{\mathbf{a}}\,. \tag{29}$$

Here $\tilde{c}_A$ and $\tilde{c}_B$ are the household's current consumption of the two goods, $\tilde{c}'_A$ and $\tilde{c}'_B$ are the household's plans for future consumption of the two goods (in every possible state of the world), $\tilde{\mathbf{a}}$ is a $k \times 1$ vector whose jth element is the household's net purchases of asset j, and $\tilde{y}_A$, $\tilde{y}_B$, $\tilde{y}'_A$ and $\tilde{y}'_B$ are the household's current and future endowments of the two goods.

The market clearing conditions for good A are

$$c_A + \tilde{c}_A = y_A + \tilde{y}_A \quad \text{and} \quad c'_A + \tilde{c}'_A = y'_A + \tilde{y}'_A\,. \tag{30}$$

When good B is tradable we have the following market clearing conditions

$$c_B + \tilde{c}_B = y_B + \tilde{y}_B \quad \text{and} \quad c'_B + \tilde{c}'_B = y'_B + \tilde{y}'_B\,. \tag{31}$$

When it is nontradable, instead, we have

$$c_B = y_B\,, \quad \tilde{c}_B = \tilde{y}_B\,, \quad c'_B = y'_B \quad \text{and} \quad \tilde{c}'_B = \tilde{y}'_B\,. \tag{32}$$

The market clearing condition in asset markets is

$$\mathbf{a} + \tilde{\mathbf{a}} = \mathbf{0}\,. \tag{33}$$

Definition. A competitive equilibrium is values of the quantities c_A, c_B, c'_A, c'_B, $\mathbf{a}$, $\tilde{c}_A$, $\tilde{c}_B$, $\tilde{c}'_A$, $\tilde{c}'_B$, $\tilde{\mathbf{a}}$ and prices, P_B, P'_B, $\tilde{P}_B$, $\tilde{P}'_B$, and $\mathbf{P_X}$ such that the quantities solve the home and foreign country optimization problems (taking the prices as given), and such that the market clearing conditions are satisfied. When good B is frictionlessly traded, Eq. (22) must also be satisfied.

2.1 Marginal Utility Growth and Risk Sharing

Our model always implies a simple relationship between the two countries' discounted marginal utility growths, or intertemporal marginal rates of substitution (IMRS), defined over aggregate consumption. We define these IMRSs as

$$m \equiv \beta u_c(c')/u_c(c) \quad \text{and} \quad \tilde{m} = \beta u_{\tilde{c}}(\tilde{c}')/u_{\tilde{c}}(\tilde{c})\,. \tag{34}$$

Similarly, we can define the two countries' IMRSs over goods A and B:

$$m_A \equiv \beta u_{c_A}(c')/u_{c_A}(c)\,, \quad \tilde{m}_A \equiv \beta u_{\tilde{c}_A}(\tilde{c}')/u_{\tilde{c}_A}(\tilde{c})\,, \tag{35}$$

$$m_B \equiv \beta u_{c_B}(c')/u_{c_B}(c)\,, \quad \tilde{m}_B \equiv \beta u_{\tilde{c}_B}(\tilde{c}')/u_{\tilde{c}_B}(\tilde{c})\,. \tag{36}$$

Definition. Perfect risk sharing describes any competitive equilibrium in which $\tilde{m}_A = m_A$ and $\tilde{m}_B = m_B$ in every possible state of the world next period.

Our definition of perfect risk sharing is the same as the one in Section 1. For any individual good, the IMRSs are equated across agents. For any identical basket of goods, suitably defined, the same is true.

As we show in the Appendix, equilibrium in the goods market always produces the intuitive result that

$$\frac{u_c(c)}{u_{c_A}(c)} = P\,, \quad \frac{u_c(c')}{u_{c_A}(c')} = P'\,, \quad \frac{u_{\tilde{c}}(\tilde{c})}{u_{\tilde{c}_A}(\tilde{c})} = \tilde{P}\,, \quad \text{and} \quad \frac{u_{\tilde{c}}(\tilde{c}')}{u_{\tilde{c}_A}(\tilde{c}')} = \tilde{P}'\,. \tag{37}$$

Combining Eq. (37) with the definitions of IMRSs in Eqs. (34) and (35), produces

$$\frac{m}{m_A} = \frac{P'}{P} \quad \text{and} \quad \frac{\tilde{m}}{\tilde{m}_A} = \frac{\tilde{P}'}{\tilde{P}}\,, \tag{38}$$

so that

$$\frac{\tilde{m}}{m} = \frac{E'}{E}\Xi\,, \quad \text{with} \quad \Xi \equiv \frac{\tilde{m}_A}{m_A}\,. \tag{39}$$

In Eq. (39), $\Xi = 1$ whenever risk sharing is perfect in frictionlessly traded goods, and $\Xi \neq 1$ when risk sharing in those goods is imperfect. For example, when asset markets are complete, agents equate IMRSs across frictionlessly traded goods, and so $m_A = \tilde{m}_A$ and therefore $\Xi = 1$. But in *any* incomplete markets setting, in general, $m_A \neq \tilde{m}_A$ and thus $\Xi \neq 1$. Also, note that Ξ is the same for any frictionlessly traded good (or basket of goods). In particular, $\tilde{m}_B/m_B = \tilde{m}_A/m_A \equiv \Xi$ whenever good B is frictionlessly traded.

2.2 Four Specific Examples

This section discusses four specific examples of our model. The four special cases we consider combine different assumptions about financial markets (complete markets vs. financial autarky) and goods market frictions (good B is frictionlessly traded vs. good B is nontraded). By explicitly solving for the equilibrium in these four cases, we demonstrate that marginal utility growths and real exchange rates are jointly determined by the laws of motion of the endowments, together with our assumptions about preferences, goods market frictions, and asset markets. We also illustrate a point we made in Section 1: The conditions under which risk sharing is imperfect, and those under which the real exchange rate varies, are different.

We adopt the assumption that $u(c) = \ln c$, and the consumption aggregates in the two countries are $c = c_A^{\theta} c_B^{1-\theta}$, and $\tilde{c} = \tilde{c}_A^{\tilde{\theta}} c_B^{1-\tilde{\theta}}$. These assumptions are useful because equilibrium prices and quantities can be worked out with pencil and paper. They imply that the CPIs in the two countries, measured in units of good A, are

$$P = \rho\, P_B^{1-\theta}\,, \qquad \text{and} \qquad \tilde{P} = \tilde{\rho}\, \tilde{P}_B^{1-\tilde{\theta}}\,, \tag{40}$$

with $\rho = \theta^{-\theta}(1-\theta)^{\theta-1}$, and $\tilde{\rho} = \tilde{\theta}^{-\tilde{\theta}}(1-\tilde{\theta})^{\tilde{\theta}-1}$. The real exchange rates in periods one and two are

$$E = (\tilde{\rho}/\rho)\tilde{P}_B^{1-\tilde{\theta}}/P_B^{1-\theta} \quad \text{and} \quad E' = (\tilde{\rho}/\rho)\tilde{P}_B'^{\,1-\tilde{\theta}}/P_B'^{\,1-\theta}\,. \tag{41}$$

We derive all of the solutions in detail in the Appendix. We use some notation in what follows. The global endowment of good A in period one is $Y_A = y_A + \tilde{y}_A$, while in period two it is $Y_A' = y_A' + \tilde{y}_A'$. Analogously, for good B we have $Y_B = y_B + \tilde{y}_B$, and $Y_B' = y_B' + \tilde{y}_B'$. The growth rates of the global endowments are $G_A = Y_A'/Y_A$ and $G_B = Y_B'/Y_B$. We also define $g_A = y_A'/y_A$, $g_B = y_B'/y_B$, $\tilde{g}_A = \tilde{y}_A'/\tilde{y}_A$ and $\tilde{g}_B = \tilde{y}_B'/\tilde{y}_B$. The home country's shares of the global endowment of good A are $s_A = y_A/Y_A$ and $s_A' = y_A'/Y_A'$, in periods one and two, respectively. Similarly, $s_B = y_B/Y_B$ and $s_B' = y_B'/Y_B'$. We let $\bar{s}_A' = \mathbb{E}[s_A']$ and $\bar{s}_B' = \mathbb{E}[s_B']$ denote the home country's average shares of the global endowments in period two.

2.2.1 Complete Markets, No Goods Market Frictions

When asset markets are complete internationally and there are no goods market frictions (i.e., good B is frictionlessly traded), then $P_B = \tilde{P}_B$ and $P_B' = \tilde{P}_B'$, and IMRSs in the individual goods are always equated across countries. As we show in the Appendix, in good A the IMRS is β/G_A. In good B the IMRS is β/G_B. Risk is shared perfectly, regardless of preferences.

In the case where preferences are identical, $E = 1$ and $E' = 1$. When preferences differ across countries the expressions in Eq. (41) simplify to $E = (\tilde{\rho}/\rho)P_B^{\theta-\tilde{\theta}}$ and $E' = (\tilde{\rho}/\rho)P_B'^{\,\theta-\tilde{\theta}}$, where $P_B = \kappa Y_A/Y_B$, $P'_B = \kappa Y'_A/Y'_B$ and

$$\kappa = \frac{(1-\tilde{\theta})(1+\beta) + (\tilde{\theta}-\theta)(s_A + \beta \bar{s}'_A)}{\tilde{\theta}(1+\beta) + (\theta - \tilde{\theta})(s_B + \beta \bar{s}'_B)} \,. \tag{42}$$

Hence,

$$\ln(E'/E) = (\theta - \tilde{\theta}) \ln(P'_B/P_B) = (\theta - \tilde{\theta}) \ln(G_A/G_B) \,. \tag{43}$$

Real exchange rate fluctuations are driven by differences in the global growth rates of the endowments of goods A and B. We see that if the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If the foreign country's preferences put more weight on good B than home country preferences (i.e.,$\tilde{\theta} < \theta$), then the foreign basket becomes relatively more expensive (the foreign country's real exchange rate appreciates).

2.2.2 Complete Markets, Good B is Nontraded

Now consider the case where asset markets are complete internationally, but good B is nontraded. In this case, in general, $P_B \neq \tilde{P}_B$. IMRSs in good A are always equated across countries: $m_A = \tilde{m}_A = \beta/G_A$. IMRSs in good B are, respectively, $m_B = \beta/g_B$ and $\tilde{m}_B = \beta/\tilde{g}_B$, so risk is not shared perfectly unless $g_B = \tilde{g}_B$ in every possible state of the world next period.

When preferences differ across countries the real exchange rates are given by Eq. (41), with prices given by

$$P_B = \kappa \frac{Y_A}{y_B} \,, \quad \tilde{P}_B = \tilde{\kappa} \frac{Y_A}{\tilde{y}_B} \,, \quad P'_B = \kappa \frac{Y'_A}{y'_B} \,, \quad \tilde{P}'_B = \tilde{\kappa} \frac{Y'_A}{\tilde{y}'_B} \,, \tag{44}$$

and

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_A + \beta \bar{s}'_A) \,, \quad \tilde{\kappa} = \frac{1-\tilde{\theta}}{(1+\beta)\tilde{\theta}} \left[1 - s_A + \beta(1-\bar{s}'_A)\right] \,. \tag{45}$$

This implies that

$$\ln(E'/E) = (1-\theta) \ln g_B - (1-\tilde{\theta}) \ln \tilde{g}_B + (\theta - \tilde{\theta}) \ln G_A \,. \tag{46}$$

Here, the real exchange rate depends on the relative growth rates of the endowment of good B in the two countries, but the two growth rates matter to different extents due to preference differences. Additionally, as was the case when good B was traded, if the foreign country's preferences put more weight on good B than home country preferences ($\tilde{\theta} < \theta$)

then, other things being equal, the foreign county's real exchange rate appreciates when the global endowment of good A grows.

If preferences are identical, then the real exchange rate in Eq. (41) simplifies to $E = (\tilde{P}_B/P_B)^{1-\theta}$ and $E' = (\tilde{P}'_B/P'_B)^{1-\theta}$ with prices still given by Eq. (44), but Eq. (45) simplified to

$$\kappa = \frac{1-\theta}{(1+\beta)\theta}(s_A + \beta \bar{s}'_A)\,, \quad \tilde{\kappa} = \frac{1-\theta}{(1+\beta)\theta}[1 - s_A + \beta(1-\bar{s}'_A)]\,. \tag{47}$$

This means that

$$\ln(E'/E) = (1-\theta)\ln(g_B/\tilde{g}_B)\,. \tag{48}$$

Here, the real exchange rate depends entirely on the relative growth rates of the endowment of good B in the two countries. If the endowment grows more slowly in the foreign country, its basket becomes relatively more expensive and its real exchange rate appreciates.

2.2.3 Financial Autarky, No Goods Market Frictions

The third case we consider is where no assets are traded internationally, but goods markets are frictionless. In this case, $P_B = \tilde{P}_B$ and $P'_B = \tilde{P}'_B$ in every possible state of the world next period. Risk sharing, in general, is imperfect. As we show in the Appendix, the ratio of IMRSs in the two countries is the same in goods A and B. That is

$$\frac{\tilde{m}_A}{m_A} = \frac{\tilde{m}_B}{m_B} = \Xi = \frac{\theta(1-s_A) + (1-\theta)(1-s_B)}{\tilde{\theta}s_A + (1-\tilde{\theta})s_B} \times \frac{\tilde{\theta}s'_A + (1-\tilde{\theta})s'_B}{\theta(1-s'_A) + (1-\theta)(1-s'_B)}. \tag{49}$$

This expression is the same when preferences are identical, except that $\theta = \tilde{\theta}$.

In the case where preferences are identical, $E = 1$ and $E' = 1$ in every possible state of the world next period. Risk sharing, on the other hand, can be good or bad. Suppose, for example, that the home country's shares of the global endowments vary and comove positively. In this case, Ξ deviates from one a lot, implying that risk sharing is limited. On the other hand, suppose that business cycles are strongly correlated across countries, so that the home country's shares of the global endowments do not change very much across different states of the world next period. In this case, Ξ will be close to one in all states, implying a high degree of risk sharing.

When preferences differ across countries then $E = (\tilde{\rho}/\rho)P_B^{\theta-\tilde{\theta}}$ and $E' = (\tilde{\rho}/\rho){P'_B}^{\theta-\tilde{\theta}}$, where $P_B = \kappa Y_A/Y_B$, $P'_B = \kappa' Y'_A/Y'_B$, and

$$\kappa = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B}\,, \qquad \kappa' = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B}\,. \tag{50}$$

Hence,

$$\ln(E'/E) = (\theta - \tilde{\theta})\left[\ln(G_A/G_B) + \ln(\kappa'/\kappa)\right]. \tag{51}$$

As in the case of complete markets, real exchange rate fluctuations are driven by differences in the growth rates of the two endowments. If the global endowment of good A grows faster than the global endowment of good B, then good B's relative price rises. If the foreign country's preferences put more weight on good B than home country preferences ($\tilde{\theta} < \theta$) then the foreign basket becomes relatively more expensive (the foreign country's real exchange rate appreciates). But the way in which the countries' shares of the global endowments fluctuate also matters for the real exchange rate. In the example we just described, the real exchange rate rises more in states of the world where $\kappa' > \kappa$. This could reflect, for example, a rise in the foreign country's share of the global endowment of good A (a drop of s'_A) at the same time as the global endowment of A rises relative to the global endowment of B.

2.2.4 Financial Autarky, Good B is Nontraded

The final case we consider combines financial autarky with the assumption that good B is nontraded. In this case, each country simply consumes its own endowments. IMRSs in the individual goods are determined by the country-specific endowment growth rates. For good A they are $m_A = \beta/g_A$ and $\tilde{m}_A = \beta/\tilde{g}_A$. In good B they are $m_B = \beta/g_B$ and $m_B = \beta/\tilde{g}_B$. Risk is not shared unless growth rates happen to coincide. The real exchange rates in the two periods are given by Eq. (41), with

$$P_B = \frac{(1-\theta)}{\theta}\frac{y_A}{y_B}, \quad \tilde{P}_B = \frac{(1-\tilde{\theta})}{\tilde{\theta}}\frac{\tilde{y}_A}{\tilde{y}_B}, \quad P'_B = \frac{(1-\theta)}{\theta}\frac{y'_A}{y'_B}, \quad \text{and} \quad \tilde{P}'_B = \frac{(1-\tilde{\theta})}{\tilde{\theta}}\frac{\tilde{y}'_A}{\tilde{y}'_B}. \tag{52}$$

Hence

$$\ln(E'/E) = (1-\tilde{\theta})\ln(\tilde{g}_A/\tilde{g}_B) - (1-\theta)\ln(g_A/g_B). \tag{53}$$

Suppose endowment growth rates are identical across goods; i.e., $g_A = g_B$ and $\tilde{g}_A = \tilde{g}_B$. Notice that this implies $E = E'$. There is no variation in the real exchange rate. The extent of risk sharing, in contrast, depends only on whether $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. It could be good or bad. Suppose, on the other hand, that risk sharing is perfect; i.e., $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$. We only get the result that $E = E'$ if $\theta = \tilde{\theta}$.

2.2.5 Discussion

Consider Table 1 from Section 1. It states that under complete markets, the observation that real exchange rates are variable only implies imperfect risk sharing when the two countries have the same consumption basket. In our model, the countries have identically-composed consumption baskets if and only if $\theta = \tilde{\theta}$, because θ and $\tilde{\theta}$ are the constant expenditure shares of good A in the two countries.

So suppose that $\theta = \tilde{\theta}$. Under complete markets, we saw that $\ln(E'/E) = 0$ and risk sharing is perfect if trade in both goods is frictionless. On the other hand, $\ln(E'/E) = (1-\theta)\ln(g_B/\tilde{g}_B)$ and $\tilde{m}_B/m_B = g_B/\tilde{g}_B$ if good B is nontraded. If one is willing to *assume* that markets are complete, *and* that countries have identical preferences, risk sharing and exchange rate changes are intimately linked in our model.

Under incomplete markets, however, there is no link, in general, between risk sharing and exchange rates, even when $\theta = \tilde{\theta}$. When $\theta = \tilde{\theta}$, and trade in both goods is frictionless, $\ln(E'/E) = 0$ yet Ξ can depart arbitrarily from one, and therefore risk sharing can be arbitrarily imperfect. When $\theta = \tilde{\theta}$, and good B is nontraded, $\ln(E'/E) = 0$ when risk sharing happens to be perfect (i.e., when $g_A = \tilde{g}_A$ and $g_B = \tilde{g}_B$ in every possible state of the world next period), but we also have $\ln(E'/E) = 0$ when risk sharing is "poor" and $g_A = g_B \neq \tilde{g}_A = \tilde{g}_B$.

More generally, our model illustrates that there is no direct link between the degree of risk sharing and real exchange rate variability.

3 Reduced Form Models

Finally, in this section we discuss the large number of papers in the recent international asset pricing literature that provide reduced form models of the domestic and foreign representative agents marginal utility growths.[13]

3.1 Complete Markets

To understand the economic motivation behind the reduced form approach in these papers, suppose that the asset returns completely span the representative agents' marginal utility growths. In this case, there are functions, m and $\tilde{m}$, of these asset returns, that are equal to Amy's and Bob's marginal utility growths,

$$m = \lambda'/\lambda \quad \text{and} \quad \tilde{m} = \tilde{\lambda}'/\tilde{\lambda}\,. \tag{54}$$

[13] A few examples of these papers include: Bansal (1997); Backus, Foresi, and Telmer (2001); Brandt and Santa-Clara (2002); Brennan and Xia (2006); Brandt, Cochrane, and Santa-Clara (2006); Bakshi, Carr, and Wu (2008); and Lustig, Roussanov, and Verdelhan (2011). It is important to note that the change in the real exchange rate is *not* the same as the real return from investing in a foreign currency (which, in turn, is not the same as the nominal return).

Also, if asset markets are complete then Amy and Bob equate indirect marginal utility growth over dollars (i.e., Eq. 6, or equivalently Eq. 7, in Section 1.1 holds) so that

$$mP/P' = \tilde{m}\tilde{P}/\tilde{P}' , \quad \text{or equivalently,} \quad E'/E = \tilde{m}/m . \tag{55}$$

Papers in this literature directly specify m and $\tilde{m}$ as reduced form functions of asset returns. They *assume* that m and $\tilde{m}$ satisfy Eq. (54) and therefore interpret these functions as Amy's and Bob's marginal utility growths, respectively. These papers assume that asset markets are complete (a necessary, but not sufficient condition for Eq. 54 to hold) and therefore they also *impose* Eq. (55).[14] Any such reduced form model obviously does not contain the necessary ingredients that we provided in Section 1.4. Therefore, Eq. (55) cannot be used to explain or understand period-by-period growth in the real exchange rate. Moreover, m and $\tilde{m}$ are functions of the asset returns, which include the growth in the exchange rate. Therefore, exchange rate growth cannot be *both* an input *and* an output of these reduced form models.[15]

There are at least a couple of reasons to question whether reduced form models of m and $\tilde{m}$ can reliably be interpreted as Amy's and Bob's marginal utility growths, respectively. First, it is impossible to test, using only asset returns, whether those returns completely span Amy's and Bob's marginal utility growths. Of course, one *can* test whether a given m and $\tilde{m}$ price the asset returns,

$$\mathbf{1} = \mathbb{E}[m\mathbf{R}P/P'] \quad \text{and} \quad \mathbf{1} = \mathbb{E}[\tilde{m}\mathbf{R}\tilde{P}/\tilde{P}'] . \tag{56}$$

However, Eq. (56) certainly does not imply Eq. (54).[16] The second reason to give pause is that, even when asset markets are complete, the set of asset returns that are used to construct m and $\tilde{m}$ in Eq. (54) must completely span agents' marginal utility growths. However, papers in this literature that provide reduced form models of m and $\tilde{m}$ only use a very small subset of the assets that are available for agents to invest in. For example, the returns in these

[14] A notable exception is Brandt and Santa-Clara (2002), which we discuss in Section 3.3 below.

[15] Brennan and Xia (2006) provide a notable exception. In the empirical section of the paper, they assume that m can be identified using only domestic government bonds and $\tilde{m}$ can be identified using only foreign government bonds. The growth in the exchange rate, E'/E, is not an input to either exercise. They empirically test Eq. (55) using the m and $\tilde{m}$ that they identify in this fashion. However, it's not clear what can be reliably concluded from this exercise. Even if asset markets are complete, so that Eq. (54) holds for some m and $\tilde{m}$, there is no general economic restriction that m depends only the returns on domestic assets and $\tilde{m}$ depends only the returns on foreign assets.

[16] Moreover, if Eq. (55) is *imposed* so that $mP/P' = \tilde{m}\tilde{P}/\tilde{P}'$, then the equations in (56) are one and the same since

$$\mathbf{1} = \mathbb{E}[m\mathbf{R}P/P'] \quad \Leftrightarrow \quad \mathbf{1} = \mathbb{E}[\tilde{m}\mathbf{R}\tilde{P}/\tilde{P}'] .$$

papers usually include nominal currency investments, as well as zero coupon bonds in two (occassionally more) countries or options on foreign currencies. Even if one accepts the assumption that asset markets are complete, one might reasonably expect that the set of assets that are required to span all agents' marginal utility growths also includes the major equity markets, other government bond markets, corporate bond markets, other foreign currencies, and perhaps even commodity and real estate markets.

3.2 Standard Approach with a Single SDF

It is instructive to compare the reduced form approach used in the recent international asset pricing literature with the standard approach to reduced form modeling of SDFs that is used in the broader asset pricing literature. Consider *any* set of dollar-denominated asset returns, $\mathbf{R}$. If there are no-arbitrage opportunites in those returns then it is well-known that we can always find a stochastic discount factor (SDF), $M > 0$, that prices them

$$\mathbf{1} = \mathbb{E}\left[\mathbf{R}M\right] . \tag{57}$$

Given any reduced form model of an SDF, $M > 0$, that satisfies Eq. (57), we can always *define*

$$m \equiv MP'/P \quad \text{and} \quad \tilde{m} \equiv M\tilde{P}'/\tilde{P}. \tag{58}$$

Eq. (55) therefore holds, by definition, with m and $\tilde{m}$ defined from M by Eq. (58).

Any reduced form model of m and $\tilde{m}$ that also *imposes* Eq. (55), is mathematically *identical* to the standard asset pricing approach of providing a reduced form model of a single SDF, $M > 0$, that satisfies Eq. (57). The only difference between these two approaches is the economic interpretation that is provided in the papers that use them. In the standard approach, M (or equivalently, $mP/P' = \tilde{m}\tilde{P}/\tilde{P}'$ when Eq. 55 is imposed) is typically interpreted as akin to the projection of all agents' indirect marginal utility growths onto the space of dollar denominated asset returns $\mathbf{R}$. There is no attempt to identify M with any single agent, since it is well-known that this projection is the same for all agents (e.g., recall Eqs. 17–19 in Section 1.3). Neither this modeling approach, nor this economic interpretation of M, requires that the asset returns completely span agents' (indirect) marginal utility growths.

By contrast, the economic interpretation of m and $\tilde{m}$ (or equivalently, MP'/P and $M\tilde{P}'/\tilde{P}$) as Amy's and Bob's marginal utility growths *does* require that the asset returns completely span agents' marginal utility growths. If that assumption does not hold, then at least one of their marginal utility growths *cannot* be written as a function of those asset returns alone. Moreover, if the asset returns do not completely span agents' marginal utility growths, then Eq. (55) does *not* necesarily hold for general m and $\tilde{m}$ that satisfy Eq. (56). In particular, recall from Section 1.2.1 that if m and $\tilde{m}$ are the projections of Amy's and Bob's marginal utility growths onto the asset returns denominated in units of their respective consumption baskets, then, in general, $E'/E \neq \tilde{m}/m$ so that Eq. (55) does *not* necessarily hold.

3.3 Incomplete Markets

Brandt and Santa-Clara (2002) provide reduced form models of m and $\tilde{m}$, which they interpet as the marginal utility growths of the domestic and foreign representative agents respectively. However, they assume that asset markets are incomplete and therefore do not impose Eq. (55). They use their model of m and $\tilde{m}$ to price foreign and domestic zero coupon bonds. For example, if m and $\tilde{m}$ both price a one-period (real) foreign zero coupon bond that costs b^* today then

$$b^* = \mathbb{E}\left[1 m E'/E\right] \quad \text{and} \quad b^* = \mathbb{E}\left[1\tilde{m}\right] . \tag{59}$$

Brandt and Santa-Clara (2002) note that Eq. (59) can be satisfied if

$$mE'/E = \tilde{m}O\,, \quad \text{or equivalently,} \quad \frac{E'}{E} = \frac{\tilde{m}}{m}O\,, \tag{60}$$

where $\mathbb{E}[O] = 1$ and O is independent of m, $\tilde{m}$, and all assets.[17] They state that "the key insight of our model is that when markets are incomplete, the volatility of the exchange rate is not uniquely determined by the domestic and foreign stochastic discount factors."

There are two problems with the reduced form approach in Brandt and Santa-Clara (2002). First, if the asset returns do not completely span the representative agents' marginal utility growths, then at least one of their marginal utility growths *cannot* be written as a function of those asset returns alone. Therefore, it is logically inconsistent to interpret reduced form models of m and $\tilde{m}$ as the marginal utility growths of the domestic and foreign representative agents, and simultaneously assume that the asset returns used to construct m and $\tilde{m}$ do not span both of their marginal utility growths. Second (and irrespective of the specific economic interpretation of m and $\tilde{m}$), the assumption in Eq. (60) that O is independent of m, $\tilde{m}$, and all assets, introduces an arbitrage opportunity into their model. In particular, both m and $\tilde{m}$ must also price a one-period (real) domestic zero coupon bond

[17] See equation 24 on page 176 in Brandt and Santa-Clara (2002).

costs b today, so that

$$b = \mathbb{E}[1m] \quad \text{and} \quad b = \mathbb{E}[1\tilde{m}E/E'] \,. \tag{61}$$

However, this additional economic restriction in Eq. (61) reveals a contradiction (internal inconsistency) in their model, since (by Jensen's inequality)

$$b = \mathbb{E}[1\tilde{m}E/E'] \equiv \mathbb{E}[1m/O] = \mathbb{E}[1m]\,\mathbb{E}[1/O] > \mathbb{E}[1m]/\mathbb{E}[O] = \mathbb{E}[1m] = b\,. \tag{62}$$

That is, the model in Brandt and Santa-Clara (2002) is not free of arbitrage opportunities, since it assigns two different prices to the same zero-coupon bond.

4 Conclusion

The recent literature in international finance has used the asset market view of real exchange rates to explain and interpret a number of empirical properties of both real and nominal exchange rates. In this paper we showed that, alone, the asset market view of real exchange rates is not useful for understanding, or giving an economic interpretation to, changes in exchange rates. Instead, we argue that in order to explain how real exchange rates are determined, it is necessary to make specific assumptions about preferences, goods market frictions, the assets agents can trade, and the nature of endowments or production.

References

Fernando Alvarez, Andrew Atkeson, and Patrick J. Kehoe. If exchange rates are random walks, then almost everything we say about monetary policy is wrong. *American Economic Review*, 97(2):339–345, September 2007.

Prakash Apte, Piet Sercu, and Raman Uppal. The exchange rate and purchasing power parity: extending the theory and tests. *Journal of International Money and Finance*, 23 (4):553–571, June 2004.

Philippe Bacchetta and Eric van Wincoop. Can information heterogeneity explain the exchange rate determination puzzle? *The American Economic Review*, 96(3):552–576, June 2006.

David K. Backus and Gregor W. Smith. Consumption and real exchange rates in dynamic economies with non-traded goods. *Journal of International Economics*, 35(3–4):297 – 316, 1993.

David K. Backus, Patrick J. Kehoe, and Finn E. Kydland. International real business cycles. *Journal of Political Economy*, 100(4):745–775, August 1992.

David K. Backus, Silverio Foresi, and Chris I. Telmer. Affine term structure models and the forward premium anomaly. *Journal of Finance*, 56(1), February 2001.

Gurdip Bakshi, Peter Carr, and Liuren Wu. Stochastic risk premiums, stochastic skewness in currency options, and stochastic discount factors in international economies. *Journal of Financial Economics*, 87(1):132–156, January 2008.

Ravi Bansal. An exploration of the forward premium puzzle in currency markets. *The Review of Financial Studies*, 10(2):369–403, Summer 1997.

Ravi Bansal and Ivan Shaliastovich. A long-run risks explanation of predictability puzzles in bond and currency markets. Working Paper, February 2010.

Marianne Baxter and Mario J. Crucini. Business cycles and the asset structure of foreign trade. *International Economic Review*, 36(4):821–854, November 1995.

Gianluca Benigno and Christoph Thoenissen. Consumption and real exchange rates with incomplete markets and non-traded goods. *Journal of International Money and Finance*, 27(6):926–948, October 2008.

Caroline Betts and Michael B. Devereux. The exchange rate in a model of pricing-to-market. *European Economic Review*, 40(3–5):1007–1021, April 1996.

Michael W. Brandt and Pedro Santa-Clara. Simulated likelihood estimation of diffusions with an application to exchange rate dynamics in incomplete markets. *Journal of Financial Economics*, 63(2):161–210, February 2002.

Michael W. Brandt, John H. Cochrane, and Pedro Santa-Clara. International risk sharing is better than you think, or exchange rates are too smooth. *Journal of Monetary Economics*, 53(4):671–698, May 2006.

Michael J. Brennan and Yihong Xia. International capital markets and foreign exchange risk. *The Review of Financial Studies*, 19(3):753–795, Fall 2006.

John Y. Campbell and John H. Cochrane. By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy*, 107(2):205–251, April 1999.

V. V. Chari, Patrick J. Kehoe, and Ellen R. McGrattan. Can sticky price models generate volatile and persistent real exchange rates? *The Review of Economic Studies*, 69(3): 533–563, July 2002.

Riccardo Colacito and Mariano M. Croce. Risks for the long run and the real exchange rate. *Journal of Political Economy*, 119(1):153–181, February 2011.

Harold L. Cole and Maurice Obstfeld. Commodity trade and international risk sharing: How much do financial markets matter? *Journal of Monetary Economics*, 28(1):3–24, August 1991.

Giancarlo Corsetti, Luca Dedola, and Sylvain Leduc. International risk sharing and the transmission of productivity shocks. *The Review of Economic Studies*, 75(2):443–473, April 2008.

Bernard Dumas. Dynamic equilibrium and the real exchange rate in a spatially separated world. *Review of Financial Studies*, 5(2):153–180, 1992.

Larry G. Epstein and Stanley E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57(4):937–969, July 1989.

Emmanuel Farhi, Samuel P. Fraiberger, Xavier Gabaix, Romain Ranciere, and Adrien Verdelhan. Crash risk in currency markets. Working Paper, 1 June 2009.

Narayana R. Kocherlakota and Luigi Pistaferri. Household heterogeneity and real exchange rates. *The Economic Journal*, 117(519):C1–C25, March 2007.

Robert E. Lucas Jr. Interest rates and currency prices in a two-country world. *Journal of Monetary Economics*, 10(3):335–359, 1982.

Hanno Lustig and Adrien Verdelhan. Investing in foreign currency is like betting on your intertemporal marginal rate of substitution. *Journal of the European Economic Association*, 4(2/3):644–655, April 2006.

Hanno Lustig and Adrien Verdelhan. The cross section of foreign currency risk premia and consumption growth risk. *The American Economic Review*, 97(1):89–117, March 2007.

Hanno N. Lustig and Adrien Verdelhan. Exchange rates in a stochastic discount factor framework. In Jessica James, Ian Marsh, and Lucio Sarno, editors, *Handbook of Exchange Rates*, Wiley Handbooks in Financial Engineering and Econometrics, chapter 14, pages 391–420. Wiley, July 2012.

Hanno N. Lustig, Nikolai L. Roussanov, and Adrien Verdelhan. Common risk factors in currency markets. *Review of Financial Studies*, 2011.

Maurice Obstfeld and Kenneth Rogoff. Exchange rate dynamics redux. *Journal of Political Economy*, 103(3):624–660, June 1995.

Anna Pavlova and Roberto Rigobon. Asset prices and exchange rates. *The Review of Financial Studies*, 20(4):1139–1181, July 2007.

Piet Sercu, Raman Uppal, and Cynthia Van Hulle. The exchange rate in the presence of transaction costs: Implications for tests of purchasing power parity. *The Journal of Finance*, 50(4):1309–1319, 1995.

Peter Smith and Michael Wickens. Asset pricing with observable stochastic discount factors. *Journal of Economic Surveys*, 16(3):397–446, 2002.

Andreas Stathopoulos. Asset Prices and Risk Sharing in Open Economies. *SSRN eLibrary*, 2011.

Alan C. Stockman. A theory of exchange rate determination. *Journal of Political Economy*, 88(4):673–698, August 1980.

Alan C. Stockman and Linda L. Tesar. Tastes and technology in a two-country model of the business cycle: Explaining international comovements. *The American Economic Review*, 85(1):168–185, March 1995.

Hal R. Varian. *Microeconomic Analysis*. W.W. Norton and Company, 2nd edition, 1984.

Adrien Verdelhan. A habit-based explanation of the exchange rate risk premium. *The Journal of Finance*, 65(1):123–146, 2010.

A Appendix

A.1 Aggregate Prices

The overall consumption aggregate for the domestic household is $c(c_A, c_B)$. Given a particular set of prices (in an arbitrary numeraire) for the individual goods, we can solve the household's static expenditure minimization problem

$$\min_{c_A, c_B} P_A c_A + P_B c_B \quad \text{subject to} \quad c = c(c_A, c_B)\,. \tag{63}$$

Because $c(\cdot)$ is a homogenous of degree one function, minimized expenditure is equal to Pc where $P = H(P_A, P_B)$. The function $H(\cdot)$ is also homogenous of degree one in its arguments, and is related to the function $c(\cdot)$ [see Varian (1984)]. The aggregate price index has the interpretation of being the Lagrange multiplier on the constraint at the optimum. To see this, notice that the first order conditions for the expenditure minimization problem are

$$P_A = \theta c_{c_A}(c_A, c_B) \quad P_B = \theta c_{c_B}(c_A, c_B)\,. \tag{64}$$

Multiplying these through these conditions by c_A and c_B and adding up you get $P_A c_A + P_B c_B = \theta c$ hence $P = \theta$. We also have

$$H(P_A, P_B) = H[P c_{c_A}(\cdot), P c_{c_B}(\cdot)] = P H[c_{c_A}(\cdot), c_{c_B}(\cdot)]\,,$$

establishing that at the optimum, $H[c_{c_A}(\cdot), c_{c_B}(\cdot)] = 1$.

Of course, a similar approach may be used for the foreign household.

A.2 Overall Marginal Utility

The asset payoffs, and all variables in period two, depend on the state of the world in period two. For concreteness, in this appendix we assume that the state of the world is indexed by $z \in \mathcal{Z} = \{1, 2, \ldots, n\}$, with n finite. The assumption that the number possible states of the world in period two is finite, or even countable, is not important and is only for ease of exposition.

By nesting the expenditure minimization problem described in Section A.1 within the domestic household's problem, we can rewrite the latter as follows. The household in the home country chooses c, $c'(z)$, and $\mathbf{a}$ to maximize

$$u(c) + \beta \sum_{z=1}^{n} u\left[c'(z)\right]\pi(z) \tag{65}$$

subject to

$$Pc + \mathbf{P_X} \cdot \mathbf{a} = y_A + P_B y_B\,, \tag{66}$$

$$P'(z)c'(z) = y'_A(z) + P'_B(z)y'_B(z) + \mathbf{X}\left(z\right) \cdot \mathbf{a}\,, \quad z = 1, \ldots, n. \tag{67}$$

The first order conditions for c, $c'(z)$, and $\mathbf{a}$ are

$$u_c(c) = P\lambda\,, \tag{68}$$

$$\beta u_c[c'(z)]\pi(z) = P'(z)\mu(z)\,, \quad z = 1, \ldots, n, \tag{69}$$

$$\mathbf{P_X}\lambda = \sum_{z=1}^{n} \mu(z)\mathbf{X}(z)\,. \tag{70}$$

Here λ is the Lagrange multiplier on the constraint (68), and $\mu(z)$ is the Lagrange multiplier on the constraint (69). So, combining (68) and (69), we get the following expression for the home household's marginal utility growth defined over its basket:

$$m(z) = \frac{\beta u_c[c'(z)]}{u_c(c)} = \frac{P'(z)}{P}\frac{\mu(z)}{\lambda\pi(z)}\,. \tag{71}$$

The household in the foreign country chooses $\tilde{c}$, $\{\tilde{c}'(z)\}_{z=1}^{n}$, and $\tilde{\mathbf{a}}$ to maximize

$$u(\tilde{c}) + \beta \sum_{z=1}^{n} u\left[\tilde{c}'(z)\right]\pi(z) \tag{72}$$

subject to

$$\tilde{P}\tilde{c} + \mathbf{P_X} \cdot \tilde{\mathbf{a}} = \tilde{y}_A + \tilde{P}_B \tilde{y}_B\,, \tag{73}$$

$$\tilde{P}'(z)\tilde{c}'(z) = \tilde{y}'_A(z) + \tilde{P}'_B(z)\tilde{y}'_B(z) + \mathbf{X}\left(z\right) \cdot \tilde{\mathbf{a}}\,, \quad z = 1, \ldots, n. \tag{74}$$

The first order conditions for $\tilde{c}$, $\{\tilde{c}'(z)\}_{z=1}^{n}$, and $\tilde{\mathbf{a}}$ are

$$u_c(\tilde{c}) = \tilde{P}\tilde{\lambda}\,, \tag{75}$$

$$\beta u_c[\tilde{c}'(z)]\pi(z) = \tilde{P}'(z)\tilde{\mu}(z)\,, \quad z = 1, \ldots, n, \tag{76}$$

$$\mathbf{P_X}\tilde{\lambda} = \sum_{z=1}^{n} \tilde{\mu}(z)\mathbf{X}(z)\,. \tag{77}$$

Here $\tilde{\lambda}$ is the Lagrange multiplier on the constraint (75), and $\tilde{\mu}(z)$ is the Lagrange multiplier on the constraint (76). So, combining (75) and (76), we get the following expression for the home household's marginal utility growth defined over its basket:

$$\tilde{m}(z) = \frac{\beta u_c[\tilde{c}'(z)]}{u_c(\tilde{c})} = \frac{\tilde{P}'(z)}{\tilde{P}}\frac{\tilde{\mu}(z)}{\tilde{\lambda}\pi(z)}\,. \tag{78}$$

Notice that m is an SDF for payoffs and prices measured in home country basket units. This is because Eqs. (70) and (71) combined imply

$$\frac{\mathbf{P_X}}{P} = \sum_{z=1}^{n} m(z)\frac{\mathbf{X}(z)}{P'(z)}\pi(z)\,. \tag{79}$$

Similarly, $\tilde{m}$ is an SDF for payoffs and prices measured in foreign country basket units. This is because Eqs. (77) and (78) combined imply

$$\frac{\mathbf{P_X}}{\tilde{P}} = \sum_{z=1}^{n} \tilde{m}(z)\frac{\mathbf{X}(z)}{\tilde{P}'(z)}\pi(z)\,. \tag{80}$$

From (71) and (78), the ratio of $\tilde{m}$ to m is

$$\frac{\tilde{m}(z)}{m(z)} = \left[\frac{\tilde{P}'(z)}{\tilde{P}}\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \Big/ \left[\frac{P'(z)}{P}\frac{\mu(z)}{\lambda}\right] = \left[\frac{E'(z)}{E}\right]\cdot\left[\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \Big/ \left[\frac{\mu(z)}{\lambda}\right]. \tag{81}$$

We define

$$\Xi(z) = \left[\frac{\tilde{\mu}(z)}{\tilde{\lambda}}\right] \Big/ \left[\frac{\mu(z)}{\lambda}\right].$$

Notice that since good A is the numeraire, the first order conditions for c_A and $\tilde{c}_A$, given in Eq. (64), along with Eqs. (68) and (75) imply that the time one marginal utilities of good A in the two countries are

$$u_c(c)c_{c_A}(c_A, c_B) = \lambda \quad u_c(\tilde{c})\tilde{c}_{c_A}(\tilde{c}_A, c_B) = \tilde{\lambda}\,. \tag{82}$$

Similarly, when these first order conditions are combined with Eqs. (69) and (76), we get expressions for the time two discounted marginal utilities of good A in the two countries:

$$\beta u_c[c'(z)]c_{c_A}[c'_A(z), c'_B(z)] = \mu(z)/\pi(z)\,, \quad z = 1, \ldots, n, \tag{83}$$

$$\beta u_c[\tilde{c}'(z)]\tilde{c}_{c_A}[\tilde{c}'_A(z), \tilde{c}'_B(z)] = \tilde{\mu}(z)/\pi(z)\,, \quad z = 1, \ldots, n. \tag{84}$$

Thus, $m_A(z) = \mu(z)/[\lambda\pi(z)]$ and $\tilde{m}_A(z) = \tilde{\mu}(z)/[\tilde{\lambda}\pi(z)]$ are the discounted marginal utility growths of good A in the two countries. Consequently,

$$\frac{\tilde{m}(z)}{m(z)} = \left[\frac{E'(z)}{E}\right] \cdot \Xi(z)\,, \tag{85}$$

with $\Xi(z) = \tilde{m}_A(z)/m_A(z)$ being a measure of risk sharing in the frictionlessly traded good (good A).

The first order conditions for c_B and $\tilde{c}_B$, given in Eq. (64), along with Eqs. (68) and (75) imply that the time one marginal utilities of good B in the two countries are

$$u_c(c)c_{c_B}(c_A, c_B) = \lambda P_B \quad u_c(\tilde{c})\tilde{c}_{c_B}(\tilde{c}_A, c_B) = \tilde{\lambda}\tilde{P}_B\,. \tag{86}$$

Similarly, when these first order conditions are combined with Eqs. (69) and (76), we get expressions for the time two discounted marginal utilities of good B in the two countries:

$$\beta u_c[c'(z)]c_{c_B}[c'_A(z), c'_B(z)] = \mu(z)P'_B(z)/\pi(z)\,, \quad z = 1, \ldots, n, \tag{87}$$

$$\beta u_c[\tilde{c}'(z)]\tilde{c}_{c_B}[\tilde{c}'_A(z), \tilde{c}'_B(z)] = \tilde{\mu}(z)\tilde{P}'_B(z)/\pi(z)\,, \quad z = 1, \ldots, n. \tag{88}$$

Thus, $m_B(z) = m_A(z)P'_B(z)/P_B$ and $\tilde{m}_B(z) = \tilde{m}_A(z)\tilde{P}'_B(z)/\tilde{P}_B$ are the discounted marginal utility growths of good B in the two countries. Consequently, $\Xi(z)[\tilde{P}'_B(z)/\tilde{P}_B]/[P'_B(z)/P_B]$ is a measure of how well risk is shared in good B. If good B is frictionlessly traded the price terms in this expression cancel out and the measure of risk sharing in good B is also $\Xi(z)$.

When the securities span variation in households' marginal utilities (i.e., if financial markets are complete) the first order conditions for a and a become equivalent to

$$\boldsymbol{\psi}\lambda = \boldsymbol{\mu}\,, \qquad \boldsymbol{\psi}\tilde{\lambda} = \tilde{\boldsymbol{\mu}}\,, \tag{89}$$

where $\boldsymbol{\mu}$ is an $n \times 1$ vector whose zth element is $\mu(z)$, $\tilde{\boldsymbol{\mu}}$ is an $n \times 1$ vector whose zth element is $\tilde{\mu}(z)$ and $\boldsymbol{\psi}$ is an $n \times 1$ vector whose zth element is $\psi(z)$, the price of a claim that pays one unit of good A in state z. Notice that when financial markets are complete, this implies $m_A(z) = \tilde{m}_A(z) = \psi(z)/\pi(z)$ and $\Xi(z) = 1$.

A.3 Equilibrium in the Special Cases

To solve the model in the special cases we assume from the start that there is a complete set of state contingent securities indexed by z. Security z pays one unit of good A in state z

and zero otherwise. It's price is $\psi(z)$ in the home country and $\tilde{\psi}(z)$ in the foreign country. If there is international trade in these assets (the complete markets case), we have $\psi(z) = \tilde{\psi}(z)$. Under financial autarky, the prices can be different.

The first order conditions for the individual consumption goods and holdings of the securities are

$$\theta c_A^{-1} = \lambda, \tag{90}$$

$$(1-\theta)c_B^{-1} = P_B\lambda, \tag{91}$$

$$\beta\theta c_A'(z)^{-1}\pi(z) = \mu(z), \quad z = 1, \ldots, n, \tag{92}$$

$$\beta(1-\theta)c_B'(z)^{-1}\pi(z) = P_B'(z)\mu(z), \quad z = 1, \ldots, n, \tag{93}$$

$$\psi(z)\lambda = \mu(z), \quad z = 1, \ldots, n. \tag{94}$$

$$\tilde{\theta}\tilde{c}_A^{-1} = \tilde{\lambda}, \tag{95}$$

$$(1-\tilde{\theta})\tilde{c}_B^{-1} = \tilde{P}_B\tilde{\lambda}, \tag{96}$$

$$\beta\tilde{\theta}\tilde{c}_A'(z)^{-1}\pi(z) = \tilde{\mu}(z), \quad z = 1, \ldots, n, \tag{97}$$

$$\beta(1-\tilde{\theta})\tilde{c}_B'(z)^{-1}\pi(z) = \tilde{P}_B'(z)\tilde{\mu}(z), \quad z = 1, \ldots, n, \tag{98}$$

$$\tilde{\psi}(z)\tilde{\lambda} = \tilde{\mu}(z), \quad z = 1, \ldots, n. \tag{99}$$

We can rewrite the first order conditions for the consumptions, using the first order conditions for the securities, as:

$$\theta = \lambda c_A \tag{100}$$

$$1 - \theta = \lambda c_B P_B \tag{101}$$

$$\beta\theta = \frac{\psi(z)\lambda}{\pi(z)} c_A'(z) \tag{102}$$

$$\beta(1-\theta) = \frac{\psi(z)\lambda}{\pi(z)} c_B'(z) P_B'(z) \tag{103}$$

$$\tilde{\theta} = \tilde{\lambda}\tilde{c}_A \tag{104}$$

$$1 - \tilde{\theta} = \tilde{\lambda}\tilde{P}_B\tilde{c}_B \tag{105}$$

$$\beta\tilde{\theta} = \frac{\tilde{\psi}(z)\tilde{\lambda}}{\pi(z)} \tilde{c}_A'(z) \tag{106}$$

$$\beta(1-\tilde{\theta}) = \frac{\tilde{\psi}(z)\tilde{\lambda}}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{107}$$

Here, we have dropped the $z = 1, \ldots, n$, from the equations for convenience.

In what follows we will use the notation $L = \lambda^{-1}$, $\tilde{L} = \tilde{\lambda}^{-1}$. From Eqs. (100), (100), (100) and (100), we see that L and $\tilde{L}$ are the households' respective total expenditures on goods in period one. We also define the global endowments: $Y_A = y_A + \tilde{y}_A$, $Y_B = y_B + \tilde{y}_B$, $Y'_A(z) = y'_A(z) + \tilde{y}'_A(z)$, $Y'_B(z) = y'_B(z) + \tilde{y}'_B(z)$. Additionally we define $G_A(z) = Y'_A(z)/Y_A$, $G_B(z) = Y'_B(z)/Y_B$, $g_A(z) = y'_A(z)/y_A$, $g_B(z) = y'_B(z)/y_B$, $\tilde{g}_A(z) = \tilde{y}'_A(z)/\tilde{y}_A$, $\tilde{g}_B(z) = \tilde{y}'_B(z)/\tilde{y}_B$, We also use the following notation

$$s_A = y_A/Y_A \quad s_B = y_B/Y_B \quad s'_A(z) = y'_A(z)/Y'_A(z) \quad s'_B(z) \equiv y'_B(z)/Y'_B(z)$$

$$\bar{s}'_A = \sum_{z=1}^{n} s'_A(z)\pi(z) \quad \bar{s}'_B = \sum_{z=1}^{n} s'_B(z)\pi(z)$$

A.3.1 When International Asset Markets are Complete

Here we have $\psi(z) = \tilde{\psi}(z)$, which allows us to rewrite the first order conditions for the consumptions as

$$\theta L = c_A \tag{108}$$

$$(1-\theta)L = c_B P_B \tag{109}$$

$$\beta\theta L = \frac{\psi(z)}{\pi(z)}c'_A(z) \tag{110}$$

$$\beta(1-\theta)L = \frac{\psi(z)}{\pi(z)}c'_B(z)P'_B(z) \tag{111}$$

$$\tilde{\theta}\tilde{L} = \tilde{c}_A \tag{112}$$

$$(1-\tilde{\theta})\tilde{L} = \tilde{P}_B\tilde{c}_B \tag{113}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\psi(z)}{\pi(z)}\tilde{c}'_A(z) \tag{114}$$

$$\beta(1-\tilde{\theta})\tilde{L} = \frac{\psi(z)}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{115}$$

The home country household's lifetime budget constraint is

$$c_A + P_B c_B + \sum_{z=1}^{n} \psi(z)\left[c'_A(z) + P'_B(z)c'_B(z)\right] = y_A + P_B y_B + \sum_{z=1}^{n} \psi(z)\left[y'_A(z) + P'_B(z)y'_B(z)\right] \tag{116}$$

From Eqs. (108), (110), (108), and (110) we see that discounted marginal utility growth in good A in the two countries are equated:

$$m_A(z) = \beta \frac{c_A}{c'_A(z)} = \frac{\psi(z)}{\pi(z)} \qquad \tilde{m}_A(z) = \beta \frac{\tilde{c}_A}{\tilde{c}'_A(z)} = \frac{\psi(z)}{\pi(z)} \tag{117}$$

From Eqs. (109), (111), (109), and (111), discounted marginal utility growths in good B are

$$m_B(z) = \beta \frac{c_B}{c'_B(z)} = \frac{\psi(z)}{\pi(z)} \frac{P'_B(z)}{P_B} \qquad \beta \frac{\tilde{c}_B}{\tilde{c}'_B(z)} = \frac{\psi(z)}{\pi(z)} \frac{\tilde{P}'_B(z)}{\tilde{P}_B} \tag{118}$$

A.3.2 When Good B is Traded

The market clearing conditions for good A are

$$c_A + \tilde{c}_A = Y_A \tag{119}$$

$$c'_A(z) + \tilde{c}'_A(z) = Y'_A(z) \tag{120}$$

$$c_B + \tilde{c}_B = Y_B \tag{121}$$

$$c'_B(z) + \tilde{c}'_B(z) = Y'_B(z) \tag{122}$$

These market clearing conditions, together with the first order conditions, (108)–(115), imply

$$\theta L + \tilde{\theta}\tilde{L} = Y_A \tag{123}$$

$$\beta(\theta L + \tilde{\theta}\tilde{L}) = \frac{\psi(z)}{\pi(z)} Y'_A(z) \tag{124}$$

$$(1-\theta)L + (1-\tilde{\theta})\tilde{L} = P_B Y_B \tag{125}$$

$$\beta[(1-\theta)L + (1-\tilde{\theta})\tilde{L}] = \frac{\psi(z)}{\pi(z)} P'_B(z) Y'_B(z) \tag{126}$$

Given a value of L we can solve the Eqs. (123) and (125) for L and P_B:

$$\tilde{L} = \frac{Y_A}{\tilde{\theta}} - \frac{\theta}{\tilde{\theta}} L \tag{127}$$

$$P_B = \frac{\frac{\tilde{\theta}-\theta}{\tilde{\theta}} L + (\frac{1-\tilde{\theta}}{\tilde{\theta}}) Y_A}{Y_B} \tag{128}$$

If you combine Eqs. (123) and (124) you get

$$m_A(z) = \beta G_A(z)^{-1} = \frac{\psi(z)}{\pi(z)} \tag{129}$$

If you combine Eqs. (125) and (126) and previous results you get

$$P_B'(z)/P_B = G_A(z)/G_B(z) \tag{130}$$

Marginal utility growth in good B is

$$m_B(z) = m_A(z)P_B'(z)/P_B = \beta G_B(z)^{-1} \tag{131}$$

Marginal utility growths are across countries in both goods (but not across goods) are equated: $m_A(z) = \tilde{m}_A(z)$ and $m_B(z) = \tilde{m}_B(z)$. This is true regardless of preferences.

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that Eqs. (128) becomes

$$P_B = \frac{1-\theta}{\theta}\frac{Y_A}{Y_B} \tag{132}$$

and Eq. (130) implies

$$P_B'(z) = \frac{1-\theta}{\theta}\frac{Y_A'(z)}{Y_B'(z)} \tag{133}$$

Since trade is frictionless and preferences are identical $E = E'(z) = 1$.

We can solve for allocations by solving for L. To do this we consider the lifetime budget constraint, (116), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_A + \beta\bar{s}_A' + \left(\frac{1-\theta}{\theta}\right)(s_B + \beta\bar{s}_B')\right]Y_A \tag{134}$$

This implies

$$L = \frac{\theta(s_A + \beta\bar{s}_A') + (1-\theta)\left(s_B + \beta\bar{s}_B'\right)}{\theta(1+\beta)}Y_A \tag{135}$$

Eq. (127) then implies that

$$\tilde{L} = \frac{\theta[(1-s_A) + \beta(1-\bar{s}_A')] + (1-\theta)[(1-s_B) + \beta(1-\bar{s}_B')]}{\theta(1+\beta)}Y_A \tag{136}$$

Different Preferences With different preferences we need to solve for L. To do this we consider the lifetime budget constraint, (116), and use the results (and notation) so far to write it as

$$(1+\beta)L = \left[s_A + \beta\bar{s}'_A + (\frac{1-\tilde{\theta}}{\tilde{\theta}})(s_B + \beta\bar{s}'_B)\right] Y_A + \frac{\tilde{\theta}-\theta}{\tilde{\theta}}(s_B + \beta\bar{s}'_B)L \tag{137}$$

This implies

$$L = \frac{\tilde{\theta}(s_A + \beta\bar{s}'_A) + (1-\tilde{\theta})\,(s_B + \beta\bar{s}'_B)}{\tilde{\theta}(1+\beta) + (\theta-\tilde{\theta})\,(s_B + \beta\bar{s}'_B)} Y_A \tag{138}$$

Eq. (127) then implies that

$$\tilde{L} = \frac{\theta[(1-s_A) + \beta(1-\bar{s}'_A)] + (1-\theta)[(1-s_B) + \beta(1-\bar{s}'_B)]}{\tilde{\theta}(1+\beta) + (\theta-\tilde{\theta})\,(s_B + \beta\bar{s}'_B)} Y_A \tag{139}$$

and (128) implies that

$$P_B = \frac{(1-\tilde{\theta})\,(1+\beta) + \left(\tilde{\theta}-\theta\right)(s_A + \beta\bar{s}'_A)}{\tilde{\theta}\,(1+\beta) + \left(\theta-\tilde{\theta}\right)(s_B + \beta\bar{s}'_B)} \frac{Y_A}{Y_B}\,, \tag{140}$$

Given Eq. (130) we have

$$P'_B(z) = \frac{G_A(z)}{G_B(z)} P_B = \frac{(1-\tilde{\theta})\,(1+\beta) + \left(\tilde{\theta}-\theta\right)(s_A + \beta\bar{s}'_A)}{\tilde{\theta}\,(1+\beta) + \left(\theta-\tilde{\theta}\right)(s_B + \beta\bar{s}'_B)} \frac{Y'_A(z)}{Y'_B(z)}\,. \tag{141}$$

Since good B is traded, $\tilde{P}_B = P_B$ and $\tilde{P}'_B(z) = P'_B(z)$ so

$$E = (\tilde{\rho}/\rho) P_B^{\theta-\tilde{\theta}} \qquad E'(z) = (\tilde{\rho}/\rho) P'_B(z)^{\theta-\tilde{\theta}}$$

But this means

$$\ln\left[E'(z)/E\right] = (\theta-\tilde{\theta})\ln\left[P'_B(z)/P_B\right] = (\theta-\tilde{\theta})\ln\left[G_A(z)/G_B(z)\right]$$

A.3.3 When Good B is Nontraded

The market clearing conditions for good A are (119) and (120). For good B they are

$$c_B = y_B\,, \quad \tilde{c}_B = \tilde{y}_B \tag{142}$$

$$c'_B(z) = y'_B(z)\,, \quad \tilde{c}'_B(z) = \tilde{y}'_B(z) \tag{143}$$

The market clearing conditions and the first order conditions together imply that Eqs. (123) and (124) hold along with

$$(1-\theta)L = P_B y_B \qquad (1-\tilde{\theta})\tilde{L} = \tilde{P}_B \tilde{y}_B \tag{144}$$

$$\beta(1-\theta)L = \frac{\psi(z)}{\pi(z)} P'_B(z) y'_B(z)\,, \quad \beta(1-\tilde{\theta})\tilde{L} = \frac{\psi(z)}{\pi(z)} \tilde{P}'_B(z)\tilde{y}'_B(z) \tag{145}$$

Given the results so far, the lifetime budget constraint of the home household, (116), becomes:

$$(1+\beta)\theta L = \kappa Y_A,$$

where $\kappa = s_A + \beta \bar{s}'_A$, and implies

$$L = \frac{\kappa}{\theta(1+\beta)} Y_A. \tag{146}$$

If we combine Eqs. (123) and (146) we have

$$\tilde{L} = \frac{\tilde{\kappa}}{\tilde{\theta}(1+\beta)} Y_A \tag{147}$$

where $\tilde{\kappa} = 1 - s_A + \beta(1 - \bar{s}'_A)$.

Combining Eqs. (123) and (124) you get

$$m_A(z) = \beta G_A(z)^{-1} = \frac{\psi(z)}{\pi(z)} \tag{148}$$

Combining Eqs. (144), (146) and (147) we have

$$P_B = \frac{(1-\theta)\kappa}{\theta(1+\beta)} \frac{Y_A}{y_B} \qquad \tilde{P}_B = \frac{(1-\tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1+\beta)} \frac{Y_A}{\tilde{y}_B}. \tag{149}$$

If we combine (144) and (145), and make use of (148) we get

$$\frac{P'_B(z)}{P_B} = \frac{G_A(z)}{g_B(z)} \qquad \frac{\tilde{P}'_B(z)}{\tilde{P}_B} = \frac{G_A(z)}{\tilde{g}_B(z)}. \tag{150}$$

Therefore, we can write

$$P'_B(z) = \frac{(1-\theta)\kappa}{\theta(1+\beta)} \frac{Y'_A(z)}{y'_B(z)} \qquad \tilde{P}'_B(z) = \frac{(1-\tilde{\theta})\tilde{\kappa}}{\tilde{\theta}(1+\beta)} \frac{Y'_A(z)}{\tilde{y}'_B(z)}. \tag{151}$$

Discounted marginal utility growth in good B is

$$m_B(z) = \frac{\psi(z)}{\pi(z)} \frac{P'_B(z)}{P_B} = \beta / g'_B(z) \qquad \tilde{m}_B(z) = \frac{\psi(z)}{\pi(z)} \frac{\tilde{P}'_B(z)}{\tilde{P}_B} = \beta / \tilde{g}_B(z) \tag{152}$$

Identical Preferences We have

$$E = \left(\tilde{P}_B / P_B\right)^{1-\theta} = \left(\frac{\tilde{\kappa}}{\kappa} \frac{y_B}{\tilde{y}_B}\right)^{1-\theta}.$$

$$E'(z) = \left(\tilde{P}'_B(z) / P'_B(z)\right)^{1-\theta} = \left[\frac{\tilde{\kappa}}{\kappa} \frac{y'_B(z)}{\tilde{y}'_B(z)}\right]^{1-\theta}.$$

And this means

$$\ln[E'(z)/E] = (1-\theta) \ln\left[g'_B(z) / \tilde{g}'_B(z)\right]$$

Different Preferences

$$E = (\frac{\tilde{\rho}}{\rho}) \frac{\tilde{P}_B^{1-\tilde{\theta}}}{P_B^{1-\theta}} = (\frac{\tilde{\rho}}{\rho}) \frac{\left[\frac{(1-\tilde{\theta})}{(1+\beta)\tilde{\theta}} \tilde{\kappa} \frac{Y_A}{\tilde{y}_B}\right]^{1-\tilde{\theta}}}{\left[\frac{(1-\theta)}{(1+\beta)\theta} \kappa \frac{Y_A}{y_B}\right]^{1-\theta}} = (\frac{\theta}{\tilde{\theta}}) \frac{\left(\tilde{\kappa} \tilde{y}_B^{-1}\right)^{1-\tilde{\theta}}}{\left(\kappa y_B^{-1}\right)^{1-\theta}} \left(\frac{Y_A}{1+\beta}\right)^{\theta-\tilde{\theta}}.$$

$$E'(z) = (\frac{\tilde{\rho}}{\rho}) \frac{\tilde{P}'_B(z)^{1-\tilde{\theta}}}{P'_B(z)^{1-\theta}} = (\frac{\theta}{\tilde{\theta}}) \frac{[\tilde{\kappa} \tilde{y}'_B(z)^{-1}]^{1-\tilde{\theta}}}{[\kappa y'_B(z)^{-1}]^{1-\theta}} \left[\frac{Y'_A(z)}{1+\beta}\right]^{\theta-\tilde{\theta}}$$

So

$$\ln[E'(z)/E] = (1-\theta) \ln g'_B(z) - (1-\tilde{\theta}) \ln \tilde{g}'_B(z) + (\theta - \tilde{\theta}) \ln G'_A(z)$$

A.3.4 Financial Autarky

Since the countries are in financial autarky, we no longer have $\psi(z) = \tilde{\psi}(z)$, so the rearranged first order conditions for the consumptions are

$$\theta L = c_A \tag{153}$$

$$(1-\theta)L = c_B P_B \tag{154}$$

$$\beta \theta L = \frac{\psi(z)}{\pi(z)} c'_A(z) \tag{155}$$

$$\beta(1-\theta)L = \frac{\psi(z)}{\pi(z)} c'_B(z) P'_B(z) \tag{156}$$

$$\tilde{\theta}\tilde{L} = \tilde{c}_A \tag{157}$$

$$(1-\tilde{\theta})\tilde{L} = \tilde{P}_B\tilde{c}_B \tag{158}$$

$$\beta\tilde{\theta}\tilde{L} = \frac{\tilde{\psi}(z)}{\pi(z)}\tilde{c}'_A(z) \tag{159}$$

$$\beta(1-\tilde{\theta})\tilde{L} = \frac{\tilde{\psi}(z)}{\pi(z)}\tilde{P}'_B(z)\tilde{c}'_B(z) \tag{160}$$

The home country's flow budget constraints must be satisfied with no asset holdings so we have

$$c_A + P_B c_B = y_A + P_B y_B \tag{161}$$

$$c'_A(z) + P'_B(z)c'_B(z) = y'_A(z) + P'_B(z)y'_B(z). \tag{162}$$

Using the first order conditions for the consumptions we get expressions for discounted marginal utility growth:

$$m_A(z) = \beta\frac{c_A}{c'_A(z)} = \frac{\psi(z)}{\pi(z)} \qquad \tilde{m}_A(z) = \beta\frac{\tilde{c}_A}{\tilde{c}'_A(z)} = \frac{\tilde{\psi}(z)}{\pi(z)} \tag{163}$$

Discounted marginal utility growth in good B is

$$m_B(z) = \beta\frac{c_B}{c'_B(z)} = \frac{\psi(z)}{\pi(z)}\frac{P'_B(z)}{P_B} \qquad \tilde{m}_B(z) = \beta\frac{\tilde{c}_B}{\tilde{c}'_B(z)} = \frac{\tilde{\psi}(z)}{\pi(z)}\frac{\tilde{P}'_B(z)}{\tilde{P}_B} \tag{164}$$

A.3.5 When Good B is Traded

The market clearing conditions for goods are Eqs. (119)–(122). The market clearing conditions and the first order conditions together imply

$$\theta L + \tilde{\theta}\tilde{L} = Y_A \tag{165}$$

$$\frac{\theta}{\psi(z)}L + \frac{\tilde{\theta}}{\tilde{\psi}(z)}\tilde{L} = \frac{1}{\beta\pi(z)}Y'_A(z) \tag{166}$$

$$(1-\theta)L + (1-\tilde{\theta})\tilde{L} = P_B Y_B \tag{167}$$

$$(1-\theta)\frac{L}{\psi(z)} + (1-\tilde{\theta})\frac{\tilde{L}}{\tilde{\psi}(z)} = \frac{1}{\beta\pi(z)}P'_B(z)Y'_B(z)\,, \quad z = 1,\ldots,n, \tag{168}$$

We can rearrange Eqs. (165) and (167) to get:

$$\tilde{L} = \frac{1}{\tilde{\tilde{\theta}}}(Y_A - \theta L) \tag{169}$$

$$P_B = \frac{1}{\tilde{\tilde{\theta}}}\frac{(\tilde{\theta} - \theta)L + (1 - \tilde{\theta})Y_A}{Y_B}\,, \tag{170}$$

We can rearrange Eqs. (166) and (168) to get:

$$\frac{\beta\pi(z)}{\tilde{\psi}(z)}\tilde{L} = \frac{1}{\tilde{\tilde{\theta}}}\left[Y'_A(z) - \theta\frac{\beta\pi(z)}{\psi(z)}L\right]. \tag{171}$$

$$P'_B(z) = \frac{1}{\tilde{\tilde{\theta}}}\frac{(\tilde{\theta} - \theta)\frac{\beta\pi(z)}{\psi(z)}L + (1 - \tilde{\theta})Y'_A(z)}{Y'_B(z)} \tag{172}$$

The flow budget constraint, (161), and Eq. (170) imply that

$$L = y_A + \frac{\frac{\tilde{\theta}-\theta}{\tilde{\tilde{\theta}}}L + (1 - \tilde{\theta})\frac{Y_A}{\tilde{\tilde{\theta}}}}{Y_B}y_B$$

or

$$L = \frac{\tilde{\theta}s_A + (1 - \tilde{\theta})s_B}{\tilde{\theta} + (\theta - \tilde{\theta})s_B}Y_A \tag{173}$$

The flow budget constraint, (162), and Eq. (172) imply that

$$\frac{\beta\pi(z)}{\psi(z)}L = \frac{\tilde{\theta}s'_A(z) + (1 - \tilde{\theta})s'_B(z)}{\tilde{\theta} + (\theta - \tilde{\theta})s'_B(z)}Y'_A(z). \tag{174}$$

Using (173) we then have

$$m_A(z) = \frac{\psi(z)}{\pi(z)} = \beta\frac{\xi_A(z)}{G_A(z)} \quad \text{with} \quad \xi_A(z) = \frac{\frac{\tilde{\theta}s_A + (1-\tilde{\theta})s_B}{\tilde{\theta}+(\theta-\tilde{\theta})s_B}}{\frac{\tilde{\theta}s'_A(z)+(1-\tilde{\theta})s'_B(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}} \tag{175}$$

Substituting (173) into (169) we get

$$\tilde{L} = \frac{\theta(1 - s_A) + (1 - \theta)\,(1 - s_B)}{\theta + (\tilde{\theta} - \theta)(1 - s_B)}Y_A \tag{176}$$

Substituting (174) into (171) we get

$$\frac{\beta\pi(z)}{\tilde{\psi}(z)}\tilde{L} = \frac{\theta[1 - s'_A(z)] + (1 - \theta)[1 - s'_B(z)]}{\theta + (\tilde{\theta} - \theta)[1 - s'_B(z)]}Y'_A(z) \tag{177}$$

Given these results, discounted marginal utility growth in good A in the foreign country is

$$\tilde{m}_A(z) = \frac{\tilde{\psi}(z)}{\pi(z)} = \beta \frac{\tilde{\xi}_A(z)}{G_A(z)} \quad \text{with} \quad \tilde{\xi}_A(z) = \frac{\frac{\theta(1-s_A)+(1-\theta)(1-s_B)}{\theta+(\tilde{\theta}-\theta)(1-s_B)}}{\frac{\theta[1-s'_A(z)]+(1-\theta)[1-s'_B(z)]}{\theta+(\tilde{\theta}-\theta)[1-s'_B(z)]}} \tag{178}$$

Substituting (173) into (170)

$$P_B = \frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B} \frac{Y_A}{Y_B} \tag{179}$$

Substituting (174) into (172)

$$P'_B(z) = \left[\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}\right] \frac{Y'_A(z)}{Y'_B(z)} \tag{180}$$

Discounted marginal utility growth in good B in the two countries is

$$m_B(z) = \beta \frac{\xi_A(z)}{G_A(z)} \frac{P'_B(z)}{P_B} = \beta \frac{\xi_A(z)}{G_B(z)} \xi_B(z) \quad \text{with} \quad \xi_B(z) = \frac{\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}}{\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s_A}{\tilde{\theta}+(\theta-\tilde{\theta})s_B}} \tag{181}$$

$$\tilde{m}_B(z) = \beta \frac{\tilde{\xi}_A(z)}{G_A(z)} \frac{P'_B(z)}{P_B} = \beta \frac{\tilde{\xi}_A(z)}{G_B(z)} \xi_B(z) \tag{182}$$

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that Eqs. (179) and (180) simplify to

$$P_B = \frac{1-\theta}{\theta} \frac{Y_A}{Y_B} \tag{183}$$

$$P'_B(z) = \frac{1-\theta}{\theta} \frac{Y'_A(z)}{Y'_B(z)} \tag{184}$$

Since both goods are frictionlessly traded and preferences are identical $E = E'(z) = 1$.

The expressions for ξ_A and $\tilde{\xi}_A$ in Eqs. (175) and (178) simplify to

$$\xi_A(z) = \frac{\theta s_A + (1-\theta)s_B}{\theta s'_A(z) + (1-\theta)s'_B(z)} \tag{185}$$

$$\tilde{\xi}_A(z) = \frac{\theta(1-s_A) + (1-\theta)(1-s_B)}{\theta[1-s'_A(z)] + (1-\theta)[1-s'_B(z)]} \tag{186}$$

The expression for ξ_B in Eq. (181) simplifies to $\xi_B(z) = 1$, implying that

$$m_B(z) = \beta \frac{\xi_A(z)}{G_B(z)} \quad \tilde{m}_B(z) = \beta \frac{\tilde{\xi}_A(z)}{G_B(z)} \tag{187}$$

The wedge between marginal utility growths in good A, good B, and in terms of aggregate consumption is

$$\tilde{m}_A(z)/m_A(z) = \tilde{m}_B(z)/m_B(z) = \tilde{m}(z)/m(z) = \tilde{\xi}_A(z)/\xi_A(z).$$

Different Preferences Given the expressions for prices, above,

$$E = (\tilde{\rho}/\rho)P_B^{\theta-\tilde{\theta}} = (\tilde{\rho}/\rho)\left(\frac{1-\tilde{\theta}+\left(\tilde{\theta}-\theta\right)s_A}{\tilde{\theta}+\left(\theta-\tilde{\theta}\right)s_B}\frac{Y_A}{Y_B}\right)^{\theta-\tilde{\theta}}$$

and

$$E'(z) = (\tilde{\rho}/\rho)P'_B(z)^{\theta-\tilde{\theta}} = (\tilde{\rho}/\rho)\left(\frac{1-\tilde{\theta}+(\tilde{\theta}-\theta)s'_A(z)}{\tilde{\theta}+(\theta-\tilde{\theta})s'_B(z)}\frac{Y'_A(z)}{Y'_B(z)}\right)^{\theta-\tilde{\theta}}$$

A.3.6 When Good B is Nontraded

Because the B good cannot be traded the goods market clearing conditions and the home household budget constraints together imply,

$$c_A = y_A\,, \quad \tilde{c}_A = \tilde{y}_A \tag{188}$$

$$c'_A(z) = y'_A(z)\,, \quad \tilde{c}'_A(z) = \tilde{y}'_A(z) \tag{189}$$

$$c_B = y_B\,, \qquad \tilde{c}_B = \tilde{y}_B\,, \tag{190}$$

$$c'_B(z) = y'_B(z)\,, \quad \tilde{c}'_B(z) = \tilde{y}'_B(z)\,, \tag{191}$$

So

$$L = y_A/\theta, \tag{192}$$

$$P_B = \frac{1-\theta}{\theta}\frac{y_A}{y_B} \tag{193}$$

$$\frac{\psi(z)}{\pi(z)} = \beta/g_A(z) \tag{194}$$

$$P'_B(z) = \frac{1-\theta}{\theta} \frac{y'_A(z)}{y'_B(z)} \tag{195}$$

$$\tilde{L} = \tilde{y}_A / \tilde{\theta} \tag{196}$$

$$\tilde{P}_B = \frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}_A}{\tilde{y}_B}, \tag{197}$$

$$\frac{\tilde{\psi}(z)}{\pi(z)} = \beta / \tilde{g}_A(z) \tag{198}$$

$$\tilde{P}'_B(z) = \frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}'_A(z)}{\tilde{y}'_B(z)} \tag{199}$$

Discounted marginal utility growths in goods A and B are

$$m_A(z) = \beta / g_A(z) \qquad \tilde{m}_A(z) = \beta / \tilde{g}_A(z)$$

$$m_B(z) = \beta / g_B(z) \qquad \tilde{m}_B(z) = \beta / \tilde{g}_B(z)$$

Identical Preferences If preferences are identical we have $\theta = \tilde{\theta}$ so that

$$E = \left(\frac{\tilde{y}_A / \tilde{y}_B}{y_A / y_B} \right)^{1-\theta} = \left(\frac{(1-s_A)/s_A}{(1-s_B)/s_B} \right)^{1-\theta}.$$

$$E'(z) = \left(\frac{\tilde{y}'_A(z) / \tilde{y}'_B(z)}{y_A(z) / y_B(z)} \right)^{1-\theta} = \left(\frac{\left[1 - s'_A(z)\right] / s'_A(z)}{\left[1 - s'_B(z)\right] / s'_B(z)} \right)^{1-\theta}.$$

Different Preferences

$$E = (\tilde{\rho}/\rho) [\frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}_A}{\tilde{y}_B}]^{1-\tilde{\theta}} / [\frac{1-\theta}{\theta} \frac{y_A}{y_B}]^{1-\theta}.$$

$$E'(z) = (\tilde{\rho}/\rho) [\frac{1-\tilde{\theta}}{\tilde{\theta}} \frac{\tilde{y}'_A(z)}{\tilde{y}'_B(z)}]^{1-\tilde{\theta}} / [\frac{1-\theta}{\theta} \frac{y'_A(z)}{y'_B(z)}]^{1-\theta}.$$